PARTIAL REGULARITY RESULTS FOR EVOLUTIONAL P–LAPLACIAN SYSTEMS WITH NATURAL GROWTH

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ABSTRACT

Let Ω be a domain in an Euclidean space R^m for $m \geq 2$, T be a positive number and set $Q = (0,T) \times \Omega$. Let p, $1 < p < \infty$. For a map $u : Q \longmapsto R^n$, $u = (u^1, \cdots, u^n)$, consider the evolutional p–Laplacian system

$$\partial_t u^i - \sum_{\alpha,\beta=1}^{m} D_\alpha \left(|Du|_g^{p-2} g^{\alpha\beta}(z,u) D_\beta u^i \right) = f^i(z,u,Du) \quad i = 1, \cdots, n, \tag{1.1}$$

where $D_\alpha = \frac{\partial}{\partial x^\alpha}$, $\alpha = 1, \cdots, m$, Du is the spatial gradient of a map u, $Du = (D_\alpha u^i)$, $|Du|_g^2$ $= \sum_{i=1}^n \sum_{\alpha,\beta=1}^m g^{\alpha\beta}(z,u) D_\alpha u^i D_\beta u^i$, f is a Carathéodory function defined on $Q \times R^n \times R^{mn}$ with values into R^n and $\left(g^{\alpha\beta}\right)$ is a symmetric matrix with measurable coefficients $g^{\alpha\beta}(z,u)$ defined on Q. We assume that there exist positive constants λ, Λ and a such that

$$\lambda |\xi|^2 \leq g^{\alpha\beta}(z,u) \xi_\alpha \xi_\beta \leq \Lambda |\xi|^2 \tag{1.2}$$

holds for all $\xi = (\xi_\alpha) \in R^m$, almost every $z \in Q$ and all $u \in R^n$ and that

$$|f(z,u,P)| \leq a|P|^p \tag{1.3}$$

is satisfied for almost every $z \in Q$, all $u \in R^n$ and $P = (P_\alpha^i) \in R^{mn}$. Moreover, we suppose that the coefficients $g^{\alpha\beta}$ are uniformly Hölder continuous functions in $Q \times R^n$ with a Hölder exponet β, $0 < \beta < 1$, and a Hölder constant $C > 0$, that is,

$$\left| g^{\alpha\beta}(z_1,u_1) - g^{\alpha\beta}(z_2,u_2) \right| \leq C \left(|t_1 - t_2|^{\frac{1}{2}} + |x_1 - x_2| \right)^\beta + C |u_1 - u_2|^\beta \tag{1.4}$$

holds for all $z_i = (t_i, x_i) \in R^{m+1}$ and $u_i \in R^n$, $i = 1,2$. Here and in what follows, the notation $|\xi|^2 = \xi \cdot \xi = \xi_\alpha^i \xi_\alpha^i$ and $|\xi|_g^2 = g^{\alpha\beta} \xi_\alpha \cdot \xi_\beta$ is used and the summation notation over repeated indices is adopted.

The typical example of such evolution systems as (1.1) is the gradient flow for the p–energy functional, defined for functions u belonging to the Sobolev space $W^{1,p}(\Omega, R^n)$,

$$E_p(u) = \int_\Omega \tfrac{1}{p} \left(g^{\alpha\beta}(x,u) D_\alpha u \cdot D_\beta u \right)^{\frac{p}{2}} dx. \tag{1.5}$$

Then the equation, which describes the gradient flow for (1.5), is (1.1) with the coefficients $g^{\alpha\beta}(x,u)$ and the lower order term

$$f(x,u,Du) = -\tfrac{1}{2} |Du|_g^{p-2} \tfrac{dg^{\alpha\beta}}{du}(z,u) D_\alpha u \cdot D_\beta u. \tag{1.6}$$

The equation (1.1) with (1.6) concerns the gradient flow for $p-$harmonic maps between smooth, compact Riemannian manifolds. In this paper, we study a Hölder regularity of a weak solution with a "small" image

$$M = |u|_{\infty,Q} \leq \frac{\lambda^{\frac{p}{2}}}{2a}. \tag{1.7}$$

A weak solution of (1.1) is defined to be a map $u \in L^{\infty}(Q, R^n) \cap L^p_{loc}\left((0,T]; W^{1,p}_{loc}(\Omega, R^n)\right)$ satisfying

$$\int_Q \left\{-u \cdot \partial_t \phi + |Du|_g^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + f(z,u,Du) \cdot \phi\right\} dz = 0 \tag{1.8}$$

for all $\phi \in C_0^{\infty}(Q, R^n)$. We call a weak solution satisfying (1.7), a small (weak) solution. A Hölder regularity of a small solution is known to hold for the case $p = 2$, which concerns a small solution of the gradient flow for harmonic maps (see [4, 7]).

Our main result is the following.

Theorem 1 *Suppose that* $\frac{m+8-\sqrt{(m+4)^2+16}}{2} < p < \infty$ *For any* α, $0 < \alpha < 1$, *there exist a closed set* $\mathcal{S} \subset Q$, *with respect to the usual parabolic metric such that a small solution is locally Hölder continuous in* $Q \setminus \mathcal{S}$ *with any exponent* α *on the parabolic metric* $|t|^{\frac{1}{p}} + |x|$ *and the Hölder constant is bounded by a positive constant depending only on* $m, p, \lambda, \Lambda, \beta$, $[g]_{\beta, Q \times B_M(0)}$, M *and* $|Du|_{p,Q}$. *Moreover, there exists a positive number* δ_0 *depending only on* m, p *and* α *such that the* $(m - \delta_0)-$*dimensional Hausdorff measure of* $\mathcal{S}$ *is zero*

$$\mathcal{H}^{m-\delta_0}(\mathcal{S}) = 0. \tag{1.9}$$

Note that Theorem 1 contains the result in the degenerte and singular cases, since $\frac{2m}{m+2} < \frac{m+8-\sqrt{(m+4)^2+16}}{2} < 2$. We also have the result on the Hölder continuity of the gradient.

Theorem 2 *Assume the hypotheses in Theorem 1. Then there exist an exponent* α_1, $0 < \alpha_1 < 1$, *depending only on* m, p *and* β *such that the gradient of a small solution is locally Hölder continuous in* $Q \setminus \mathcal{S}$ *with an exponent* α_1 *on the usual parabolic metric. The Hölder constant is bounded by a positive constant having the same dependence as the one in Theorem 1.*

To prove Theorem, we use the perturbation argument with the homogeneous $p-$Laplaci -an system with constant coefficients. One of the crucial estimates is Gehring's Reverse Hölder inequality available for evolutional $p-$Laplacian systems, originally developed in [5]. Unfortunately, the inequality obtained in [5] can not be used in our estimates. We make modification of the cube decomposition and selection argument in [5] so that the resulting Gehring's Reverse Hölder inequality is well-worked in the perturbation estimate. We also adopt the local Hölder estimate in [6], which is a refinement of the argument of [1] (also refer to [2]).

Let α_0 and β_0 be positive numbers such that $\alpha_0 = p\left(\frac{m}{2} + 1\right) - m$ and

$$\beta_0 = \begin{cases} 2, & \text{if } p > 2, \\ 4 - p, & \text{if } \frac{2m}{m+2} < p < 2. \end{cases}$$

Set $Q_{\tau,\rho}(z_0) = (t_0-\tau, t_0)\times B_\rho(x_0)$ for $\rho, \tau > 0$ and $z_0 = (t_0, x_0) \in R^{m+1}$. Then the following inequality holds:

Lemma 3 (Gehring's Reverse Hölder inequality) *Let u be a weak solution of (1.1) with $p > \frac{2m}{m+2}$. Then there exist positive constants ϵ and C depending only on $m, p, a, \lambda, \Lambda$ and M such that*

$$\begin{aligned} \frac{1}{|Q_{\rho^p,\rho}|}\int_{Q_{\rho^p,\rho}(z_0)} |Du|^{p+\epsilon}dz \;\leq\; & C\rho^{\epsilon\left(\frac{p}{\beta_0}-1\right)}\left(\frac{1}{|Q_{(2\rho)^p,2\rho}|}\int_{Q_{(2\rho)^p,2\rho}(z_0)}|Du|^p dz\right)^{1+\frac{\epsilon}{\beta_0}} \\ & + C\rho^{\epsilon\left(\frac{p}{\alpha_0}-1\right)}\left(\frac{1}{|Q_{(2\rho)^p,2\rho}|}\int_{Q_{(2\rho)^p,2\rho}(z_0)}|Du|^p dz\right)^{1+\frac{\epsilon}{\alpha_0}} \end{aligned} \tag{1.10}$$

holds for all $Q_{\rho^p,\rho}(z_0) \subset Q_{(2\rho)^p,2\rho}(z_0) \subset Q$ with $\rho > 0$.

Note that Lemma 3 implies the higher integrability of the gradient, $Du \in L^{p+\epsilon}_{\rm loc}(Q)$. This fact essentially comes from the smoothing effect of the evolutional $p-$Laplace operator.

References

[1] Y.-Z. Chen, E. DiBenedetto, Boundary estimates for solutions of nonlinear degenerate parabolic systems, J. reine angew. Math. **395** (1989) 102-131 .

[2] E. DiBenedetto, Degenerate Parabolic Equations, Universitext, Springer-Verlag (1994).

[3] M. Giaquinta, Multiple integrals in the calculus of variations and nonlinear elliptic systems Ann. of Math. Stud. **105**, Princeton University Press, Princeton (1983).

[4] M. Giaquinta, M. Struwe, On the partial regularity of weak solutions of nonlinear parabolic systems, Math. Z. **179**, (1982) 437-451

[5] J. Kinuunen, J. L. Lewis, Higher integrability for parabolic systems of $p-$Laplacian type, Duke Math. J. **102(2)** (2000) 253-271.

[6] M. Misawa, Local Hölder regularity of gradients for evolutional $p-$Laplacian systems, preprint (2000).

[7] M. Struwe, On the Hölder continuity of bounded weak solutions of quasilinear parabolic systems, Manusc. Math. **35**, (1981) 125-145