# Nonrelativistic limit of Cauchy and scattering problems for nonlinear Klein-Gordon equations [1]

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In this talk, we study the asymptotic behavior of solutions for the nonlinear Klein-Gordon equation:

$$\ddot{u}/c^2 - \Delta u + c^2 u + f(u) = 0, \tag{1}$$

as the speed of light $c \to \infty$, where $u = u(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}$ and $f(u) = \lambda |u|^p u$ with $p > 0$ and $\lambda \in \mathbb{R}$. To eliminate the time oscillation we put $v := e^{-ic^2 t} u$, which obeys the following modulated equation:

$$\ddot{v}/c^2 + 2i\dot{v} - \Delta v + f(v) = 0. \tag{2}$$

Then we can take the singular limit as $c \to \infty$ to the nonlinear Schrödinger equation:

$$2i\dot{v} - \Delta v + f(v) = 0. \tag{3}$$

First we consider the Cauchy problem for (2) with any initial data of finite energy, which is wellposed as proved in [3]. We can prove that any solution converges in the energy space locally uniformly in time.

**Theorem 1** ([6]). *Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $p < 4/(n-2)$. Let $(\varphi_c, \psi_c) \in H^1 \oplus L^2$, $\varphi \in H^1$, $u_c$ be the solution of (2) with $(u_c(0), \dot{u}_c(0)) = (\varphi_c, \psi_c)$, and $v$ be the solution of (3) with $v(0) = \varphi$. Denote by $T_c^*$ and $T^*$ the maximal existence time of $u_c$ and $v$, respectively. Assume that*

$$(\varphi_c, \psi_c/c) \to (\varphi, 0) \quad \text{in } H^1 \oplus L^2, \tag{4}$$

*as $c \to \infty$. Then we have*

$$\liminf_{c \to \infty} T_c^* \geq T^*, \tag{5}$$

*and $u_c$ converges to $v$ in $C([0,T^*); H^1)$.*

The condition on $\psi_c$ in (4) is required for convergence of the energy of $u_c$, and it kills the other possible time oscillation of the form $e^{-ic^2 t}$ in the solution for (1). The above result considerably improves those in [10, 7, 5].

The nonrelativistic limit can not approximate solutions in $H^1$ globally in time. But if we consider the aymptotic behavior for large time through the wave operators, we can use the nonrelativistic approximation.

[1] A part of this talk concerns a collaborating work with Shuji Machihara and Tohru Ozawa, Hokkaido University.

**Definition 1.** The wave operators $W_\pm^c$ for (2) are maps from $H^1 \oplus L^2$ into itself which map the initial data $(v(0), \dot{v}(0))$ of any finite energy solution $v$ of the free modulated Klein-Gordon:

$$\ddot{v}/c^2 + 2i\dot{v} - \Delta v = 0, \tag{6}$$

into the initial data $(u_\pm(0), \dot{u}_\pm(0))$ of the solution of (2) satisfying

$$\lim_{t \to \pm\infty} \|u_\pm(t) - v(t)\|_{H^1} + \|\dot{u}_\pm(t) - \dot{v}(t)\|_{L^2} = 0, \tag{7}$$

respectively. The wave operators $W_\pm$ for (3) are defined similarly as maps from $H^1$ into itself. We denote $M_*^* := (W_*^*)^{-1}$, $S^c := M_+^c W_-^c$ and $S := M_+ W_-$.

Unique existence of these maps was proved in [2, 4, 8]. We have the following result for the scattering.

**Theorem 2** ([9]). *Assume $n \in \mathbb{N}$, $\lambda \geq 0$ and $4/n < p < 4/(n-2)$. Let $(\varphi_c, \psi_c) \in H^1 \oplus L^2$ and $\varphi \in H^1$. Suppose*

$$(\varphi_c, \psi_c/c) \to (\varphi, 0) \quad \text{in } H^1 \oplus L^2, \tag{8}$$

*as $c \to \infty$. Then we have*

$$\begin{aligned} &W_\pm^c(\varphi_c, \psi_c/c) \to (W_\pm\varphi, 0) \quad \text{in } H^1 \oplus L^2, \\ &M_\pm^c(\varphi_c, \psi_c/c) \to (M_\pm\varphi, 0) \quad \text{in } H^1 \oplus L^2, \\ &S^c(\varphi_c, \psi_c/c) \to (S\varphi, 0) \quad \text{in } H^1 \oplus L^2. \end{aligned} \tag{9}$$

The former Theorem 1 can be recovered from this Theorem 2 if $4/n < p$ and $\lambda \geq 0$, since the nonlinear flow map $U_N^c(t)$ for (1) can be written as

$$U_N^c(t) = W_\pm^c U_F^c(t) M_\pm^c, \tag{10}$$

where $U_F^c(t)$ denotes the free propagator for the modulated Klein-Gordon.

Key ingredients in our proof are the compactness argument combined with the conservation laws and the uniform estimates in Strichartz-type norms. More precisely, we use the space-time norms of the following form:

$$\|u\|_{S \mid W \cap K} := \|\chi^c * u\|_S + \|\chi_c * u\|_{W \cap K}, \tag{11}$$

where $\chi^c$ smoothly cuts off the higher frequency part $|\xi| \gtrsim c$, which is carried by the latter term $\chi_c * u = u - \chi^c * u$. $S$, $W$ and $K$ denote the space-time norms of Strichartz type for Schrödinger, wave and Klein-Gordon equations, respectively. Thus we treat the lower frequency $|\xi| \lesssim c$ of solutions as those of Schrödinger, and the higher as of Klein-Gordon. It is a natural strategy in view of the relativistic correction of speed. For example, we can choose as $S = L_t^{2+4/n} B^1_{2+4/n,2}$, $W = c^{-2-4/(n-1)} L_t^{2+4/(n-1)} B^{1/2}_{2+4/(n-1),2}$ and $K = c^{-1/2} L_t^{2+4/n} B^{1/2}_{2+4/n,2}$ if $n \geq 3$. Here $B^*_{*,2}$ denotes the inhomogeneous Besov space on $\mathbb{R}^n$ for $x$ variable.

By the Strichartz estimates adjusted to the above norms, we get a uniform closed estimate for those norms and $H^1$, like

$$\|v - v_0\|_{L^\infty H^1 \cap (S\,|\,W \cap K)} \le C T^\alpha \|v\|^{p+1}_{L^\infty H^1 \cap (S\,|\,W \cap K)}, \tag{12}$$

where $v$ is any solution to (2), $v_0$ is the free solution with the same initial data, $\alpha > 0$ is a certain constant depending on $n$ and $p$, and the norms are taken on a finite time interval $[0, T]$. This implies the uniform solvability $\liminf T_c^* \ge T^*$, and uniform $H^1$ bound on any closed interval in $[0, T^*)$. Then, by the compactness argument and the conservation of the charge and energy, we obtain the strong convergence in $H^1$. In the repulsive case $\lambda \ge 0$, we do not need the unifrom Strichartz estimate for Theorem 1, since any solution is bounded in $H^1$ by the energy and so extends globally.

The proof of Theorem 2 required us in addition a uniform decay estimate in terms of the space-time norms, so that free solutions can approximate nonlinear solutions for large time, say $t > T$, uniformly in $c$. Then we have only to make $c$ so large that the nonrelativistic limit can approximate the solutions on $[0, T]$.

The scattering result (Theorem 2) reflects the fact that the nonrelativistic limit converges globally in the space-time norms. It is also possible to retrace the argument in [8] to derive a uniform global estimate for the space-time norms in terms of the energy only. It would be interesting if we could get the same result in the Sobolev critical case $p = 4/(n-2)$, where we know only the estimate dependent on $c$ and global wellposedness for the limit equation (3) with general data is still open (see [1] for radial data).

## References


1. J. Bourgain, *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. **12** (1999), no. 1, 145–171.
2. P. Brenner, *On scattering and everywhere defined scattering operators for nonlinear Klein-Gordon equations*, J. Differential Equations **56**, (1985), no. 3, 310–344.
3. J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z. **189** (1985), 487–505.
4. J. Ginibre and G. Velo, *Scattering theory in the energy space for a class of non-linear Schrödinger equations*, J. Math. Pures Appl. **64**, (1985), 363–401.
5. S. Machihara, *The nonrelativistic limit of the nonlinear Klein-Gordon equation*, to appear in Funkcial. Ekvac.
6. S. Machihara, K. Nakanishi and T. Ozawa, *Nonrelativistic limit in the energy space of nonlinear Klein-Gordon equations*, preprint.
7. B. Najman, *The nonrelativistic limit of the nonlinear Klein-Gordon equation*, Nonlinear Anal. **15** (1990), 217–228.
8. K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2*, J. Funct. Anal. **169** (1999), no. 1, 201–225.
9. K. Nakanishi, *Nonrelativistic limit of scattering theory for nonlinear Klein-Gordon equations*, preprint.
10. M. Tsutsumi, *Nonrelativistic approximation of nonlinear Klein-Gordon equations in two space dimensions*, Nonlinear Anal. **8** (1984), 637–643.