ON THE QUADRATIC NONLINEAR SCHRODINGER EQUATION

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We study the global in time existence of small solutions to the Cauchy problem for the quadratic nonlinear Schrödinger equation in one space dimension

$$\begin{cases} iu_t + u_{xx} = (\overline{u}_x)^2, \ (t,x) \in \mathbf{R} \times \mathbf{R}, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{0.1}$$

The global in time existence of small solutions to the Cauchy problem for the quadratic nonlinear Schrödinger equation in the case of one space dimension is a difficult problem , since from the heuristic point of view the linear part of equation (0.1) decays faster than the nonlinear term, hence the nonlinearity should be considered as subcritical. However as we will show below the nonlinearity in equation (0.1) appears to be critical for large time due to it's oscillating structure. Another difficulty in our problem is that the nonlinearity does not posses a self-conjugate structure, therefore we can not estimate the operator $\mathcal{J} = x + 2it\partial_x$ directly from the equation (0.1). As we know the estimate involving the operator $\mathcal{J}$ plays a crucial role in the large time asymptotic behavior of solutions. To overcome these obstacles we combine the method of normal forms by Shatah [6] and the method developed in the previous work [3]. The method of normal forms was used to study a global existence in time of small solutions to quadratic nonlinear Klein-Gordon equations in three space dimensions [6] and in two space dimensions [4]. In paper [1], Cohn used the method of normal forms to study (0.1) and showed that the existence time T is bounded from below by $C\varepsilon^{-6}$, where ε is the size of the data in some Sobolev norm, see also [2] in the case of two dimensions. In paper [5] the case u_x^2 was studied by the so-called Hopf-Cole transformation.

We denote the linear Schrödinger evolution group

$$\mathcal{U}(t)\phi = \frac{1}{\sqrt{4\pi it}} \int e^{\frac{i}{4t}(x-y)^2} \phi(y) dy = \mathcal{F}^{-1} e^{it\xi^2} \mathcal{F}\phi = M(t) \mathcal{D}(t) \mathcal{F} M(t) \phi,$$

where $M(t) = e^{\frac{ix^2}{4t}}$, $\mathcal{D}(t)$ is the dilation operator $\mathcal{D}(t)\phi = \frac{1}{\sqrt{2it}} \phi\left(\frac{\cdot}{2t}\right)$, $\mathcal{F}\phi \equiv \hat{\phi} = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} \phi(x)\, dx$ denotes the Fourier transform of the function ϕ. Then the inverse Fourier transformation is $\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi} \phi(\xi) d\xi$, $\mathcal{D}(t)^{-1}\phi = i\mathcal{D}\left(\frac{1}{4t}\right)\phi$ and

$$\mathcal{U}(-t)\phi = i\overline{M}(t) \mathcal{F}^{-1} \mathcal{D}\left(\frac{1}{4t}\right) \overline{M}(t).$$

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We essentially use the estimates of the operators $\mathcal{J} = x + 2it\partial_x = \mathcal{U}(t) x \mathcal{U}(-t) = 2itM(t)\partial_x \overline{M}(t)$ and $\mathcal{I} = x\partial_x + 2t\partial_t$. We have the relation $\mathcal{J}\partial_x = \mathcal{I} + 2it\mathcal{L}$, where $\mathcal{L} = i\partial_t + \partial_x^2$.

We denote the usual Lebesgue space $\mathbf{L}^p = \left\{\phi \in \mathbf{S}'; \|\phi\|_p < \infty\right\}$, where the norm $\|\phi\|_p = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{1/p}$ if $1 \le p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}\{|\phi(x)|; x \in \mathbf{R}\}$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2$. Weighted Sobolev space is $\mathbf{H}_p^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{m,k,p} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\|_p < \infty\right\}$, $m, k \in \mathbf{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$. We denote also for simplicity $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$ and the norm $\|\phi\|_{m,k} = \|\phi\|_{m,k,2}$. Different positive constants we denote by the same letter C.

Now we state the result of this paper.

Theorem 0.1. *Let the initial data $u_0 \in \mathbf{H}^{3,1}$ and the norm $\|u_0\|_{3,1}$ be sufficiently small. Then there exists a unique global solution u of the Cauchy problem (0.1) such that $u \in \mathbf{C}\left(\mathbf{R}; \mathbf{H}^{3,0}\right)$. Moreover there exists a unique function $Q \in \mathbf{L}^\infty$ such that the following asymptotics is valid uniformly with respect to $x \in \mathbf{R}$*

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{t}} Q\left(\frac{x}{2t}\right) \exp\left(\frac{ix^2}{4t} + \frac{i}{4\sqrt{3}\pi}\left(\frac{x}{2t}\right)^2 \left|Q\left(\frac{x}{2t}\right)\right|^2 \log t\right) \\ &\quad + O\left(t^{-\frac{1}{2}-\gamma}\right), \end{aligned} \tag{0.2}$$

where $\gamma \in \left(0, \frac{1}{8}\right)$.

Theorem 0.2. *We assume that the conditions of Theorem 0.1 are satisfied and there exists a final state $u^+ \in \mathbf{H}^{2,\delta}$, $\delta > \frac{1}{2}$ such that*

$$\left\|u(t) - \mathcal{U}(t) u^+\right\| \to 0 \text{ as } t \to \infty.$$

Then the final state u^+ is identically zero.

One of the main points of this paper is to transform the original equation (0.1) with quadratic nonlinearity to a nonlinear Schrödinger equation with a critical cubic nonlinearity

$$\mathcal{L}\left(u - \mathcal{G}\left(\overline{u}, \overline{u}\right)\right) = 2\mathcal{G}\left(\overline{u}, (u_x)^2\right), \tag{0.3}$$

where $\mathcal{L} = i\partial_t + \partial_x^2$ and the symmetric bilinear operator

$$\mathcal{G}(\phi, \psi) = \iint g(y,z)\, \phi_x(x-y)\, \psi_x(x-z)\, dydz$$

with

$$g(y,z) = -\frac{1}{8\pi^2} \iint \frac{e^{iyq+izr}}{q^2 + r^2 + qr} dqdr.$$

Then we apply the method developed in [3] to (0.3).

Remark 0.1. *Our proof of Theorem 0.1 can be applied to the case*

$$\mathcal{L}u = \lambda\left(\overline{u}_x\right)^2 + \mu u_x^2, \text{ where } \lambda \in \mathbf{R} \text{ and } \mu \in \mathbf{C} \tag{0.4}$$

since we have

$$\begin{aligned} &\mathcal{L}\left(u - \frac{\mu}{2}u^2 - \lambda\mathcal{G}\left(\overline{u}, \overline{u}\right)\right) = -\mu u(\mathcal{L}u) + 2\lambda\mathcal{G}\left(\overline{\mathcal{L}u}, \overline{u}\right) \\ = \; &-\lambda\mu u\left(\overline{u}_x\right)^2 - \lambda\mu u u_x^2 + 2\lambda^2\mathcal{G}\left(\overline{u}, u_x^2\right) + 2\lambda\mu\mathcal{G}\left(\overline{u}, \overline{u}_x^2\right). \end{aligned}$$

The third term of the right-hand side of the above equation is treated in this paper. The rest terms are handled in the same way. When we consider the first term, we encounter the derivative loss difficulty. However we overcome this obstacle by using the smoothing property of Schrödinger evolution group. Hence in the case of equation (0.4) we have the same result as that of Theorem 0.1.

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