Instability of stationary states for nonlinear Schrödinger equations with magnetic field

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1. Introduction

We consider the nonlinear Schrödinger equation with a constant magnetic field $B = (0,0,b)$:

$$i\partial_t u = -(\nabla + iA(x))^2 u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+3}. \tag{NLS}$$

Here, $u = u(t,x)$ is a complex valued unknown function, $1 < p < 5$ and

$$A(x_1,x_2,x_3) = \frac{b}{2}(-x_2, x_1, 0), \quad b \in \mathbb{R} \setminus \{0\}.$$

We note that $B = \operatorname{rot} A(x) = (0,0,b)$, $\operatorname{div} A(x) = 0$ and

$$-(\nabla + iA(x))^2 u = -\Delta u - 2iA(x)\cdot\nabla u + |A(x)|^2 u = -\Delta u - bi\frac{\partial u}{\partial \theta} + \frac{b^2}{4}\rho^2 u,$$

where we used the cylindric coordinates (ρ,θ,z) in $\mathbb{R}^3$:

$$x_1 = \rho\cos\theta, \quad x_2 = \rho\sin\theta, \quad x_3 = z.$$

The Cauchy problem in the energy space

$$H^1_A(\mathbb{R}^3) = \{v \in L^2(\mathbb{R}^3) : \ (\nabla + iA(x))v \in L^2(\mathbb{R}^3)\}$$

was studied by Cazenave and Esteban [2]. It is proved in [2] that for any $u_0 \in H^1_A(\mathbb{R}^3)$ there exist $T > 0$ and a unique solution $u(t) \in C([0,T], H^1_A(\mathbb{R}^3))$ of (NLS) with $u(0) = u_0$, and the conservation of energy and charge holds:

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \quad t \in [0,T],$$

where E is the energy functional defined on $H^1_A(\mathbb{R}^3)$ by

$$E(v) := \frac{1}{2}\|(\nabla + iA)v\|_{L^2}^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}.$$

The stationary problem:

$$-(\nabla + iA(x))^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^3 \tag{SP}$$

with $\omega \in \mathbb{R}$ was studied by Esteban and Lions [4]. We note that if $\phi_\omega(x)$ is a solution of (SP), then $e^{i\omega t}\phi_\omega(x)$ is a solution of (NLS). Among other things, it is proved in [4] that for any $1 < p < 5$ and $k \in \mathbb{Z}$, if $\omega \in (\min\{-|b|, kb\}, \infty)$, then there exists a minimal action solution $\phi_\omega(x)$ of (SP) in $H^1_{A,k}(\mathbb{R}^3)$ (see Proposition 1 below for $k = 0$). Here, we put

$$H^1_{A,k}(\mathbb{R}^3) = \{v \in H^1_A(\mathbb{R}^3) : \ e^{ik\theta}v(\rho,\theta,z) \text{ depends only on } (\rho,z)\}.$$

The stability of stationary states was studied by Cazenave and Esteban [2], for the case $1 < p < 1 + 4/3$, by adapting the variational argument in Cazenave and Lions [3] for the case $A(x) \equiv 0$. In [6], Gonçalves Ribeiro studied the instability of minimal action stationary states $e^{i\omega t}\phi_\omega(\rho, z)$ in $H^1_{A,0}(\mathbb{R}^3)$, and he proved that if $1 + 4/3 + (4\sqrt{10} - 8)/9 \le p < 5$, then $e^{i\omega t}\phi_\omega(\rho, z)$ is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for any $\omega > 0$.

In this talk, we consider the instability of minimal action stationary states $e^{i\omega t}\phi_\omega(\rho, z)$ in $H^1_{A,0}(\mathbb{R}^3)$ for the case $1 + 4/3 < p < 5$, and we will prove that if $1 + 4/3 < p < 5$, then $e^{i\omega t}\phi_\omega(\rho, z)$ is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for sufficiently large $\omega > 0$.

Finally, we recall some known results for the case $A(x) \equiv 0$. That is, we consider the pure power nonlinear Schrödinger equation:

$$iu_t = -\Delta u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R}^{1+n}, \tag{NLS0}$$

and the corresponding stationary problem:

$$-\Delta\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n, \tag{SP0}$$

where $n \ge 1$, $p > 1$ and $p < 1 + 4/(n-2)$ if $n \ge 3$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of (SP0) in $H^1(\mathbb{R}^n)$, and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (NLS0) is stable for any $\omega > 0$ if $p < 1 + 4/n$, and unstable for any $\omega > 0$ if $p \ge 1 + 4/n$ (see, e.g., [1, 3, 7, 9, 10]). Therefore, we see that $p = 1 + 4/n$ is the critical power for the stability and instability of standing waves for (NLS0).

2. Main Result

First, we note that $A(x) \cdot \nabla v = (b/2)\partial v/\partial\theta = 0$ for $v \in H^1_{A,0}(\mathbb{R}^3)$, and

$$H^1_{A,0}(\mathbb{R}^3) = \{v \in H^1(\mathbb{R}^3) : \ \rho v \in L^2(\mathbb{R}^3), \ v = v(\rho, z) \text{ does not depend on } \theta\},$$

$$\|(\nabla + iA)v\|^2_{L^2} = \|\nabla v\|^2_{L^2} + \frac{b^2}{4}\|\rho v\|^2_{L^2}, \quad v \in H^1_{A,0}(\mathbb{R}^3).$$

We define two functionals on $H^1_{A,0}(\mathbb{R}^3)$:

$$S_\omega(v) := E(v) + \frac{\omega}{2}\|v\|^2_{L^2} = \frac{1}{2}\|\nabla v\|^2_{L^2} + \frac{b^2}{8}\|\rho v\|^2_{L^2} + \frac{\omega}{2}\|v\|^2_{L^2} - \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}},$$

$$I_\omega(v) := \langle S'_\omega(v), v\rangle = \|\nabla v\|^2_{L^2} + \frac{b^2}{4}\|\rho v\|^2_{L^2} + \omega\|v\|^2_{L^2} - \|v\|^{p+1}_{L^{p+1}}.$$

Proposition 1. (Existence of mimimal action solutions of (SP), [4]) Let $1 < p < 5$ and $\omega \in (-|b|, \infty)$. Then there exists a minimizer $\phi_\omega(\rho, z)$ of

$$\inf\{S_\omega(v) : \ v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}, \ I_\omega(v) = 0\},$$

and $\phi_\omega(\rho, z)$ satisfies (SP) in the dual space of $H^1_{A,0}(\mathbb{R}^3)$, i.e., $S'_\omega(\phi_\omega) = 0$.

Definition. We say that the stationary state solution $e^{i\omega t}\phi_\omega(\rho, z)$ of (NLS) is stable in $H^1_{A,0}(\mathbb{R}^3)$ if for any $\varepsilon > 0$ there exists $\delta > 0$ with the following property: if $u_0 \in H^1_{A,0}(\mathbb{R}^3)$ satisfies

$$\inf_{\alpha,\beta\in\mathbb{R}} \|u_0 - e^{i\alpha}\phi_\omega(\cdot, \cdot + \beta)\|_{H^1_A} < \delta,$$

then the solution $u(t)$ of (NLS) with $u(0) = u_0$ exists for all $t \geq 0$ and satisfies

$$\sup_{t\geq 0} \inf_{\alpha,\beta\in\mathbb{R}} \|u(t) - e^{i\alpha}\phi_\omega(\cdot, \cdot + \beta)\|_{H^1_A} < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega(\rho, z)$ is said to be unstable in $H^1_{A,0}(\mathbb{R}^3)$.

Our main result in this talk is the following.

Theorem 1. Let $1 + 4/3 < p < 5$ and $\phi_\omega(\rho, z)$ be the minimal action solution of (SP) in $H^1_{A,0}(\mathbb{R}^3)$ constructed in Proposition 1. Then there exists $\omega_* \in (0, \infty)$ such that $e^{i\omega t}\phi_\omega(\rho, z)$ is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for any $\omega \in (\omega_*, \infty)$.

To prove Theorem 1, we use the following sufficient condition for instability.

Proposition 2. ([6, 8]) Let $\phi_\omega(\rho, z)$ be the minimal action solution of (SP) in $H^1_{A,0}(\mathbb{R}^3)$ constructed in Proposition 1. If $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} < 0$, then $e^{i\omega t}\phi_\omega(x)$ is unstable in $H^1_{A,0}(\mathbb{R}^3)$. Here, $v^\lambda(x) := \lambda^{n/2}v(\lambda x)$ for $\lambda > 0$.

Remark. Since $\|v^\lambda\|^2_{L^2} = \|v\|^2_{L^2}$ and $\partial_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = \partial_\lambda S_\omega(\phi^\lambda_\omega)|_{\lambda=1} = 0$, the assumption $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} < 0$ implies that $\phi_\omega(x)$ is not a local minimizer of the energy functional E on $\{v \in H^1_{A,0}(\mathbb{R}^3) : \|v\|^2_{L^2} = \|\phi_\omega\|^2_{L^2}\}$.

3. Outline of proof for Theorem 1

In this section, to prove Theorem 1, we check the condition $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} < 0$ for the minimal action solution $\phi_\omega(\rho, z)$ of (SP) constructed in Proposition 1. By simple computations, we have

$$E(\phi^\lambda_\omega) = \frac{\lambda^2}{2}\|\nabla\phi_\omega\|^2_{L^2} + \frac{\lambda^{-2}b^2}{8}\|\rho\phi_\omega\|^2_{L^2} - \frac{\lambda^{3(p-1)/2}}{p+1}\|\phi_\omega\|^{p+1}_{L^{p+1}},$$

$$\partial_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = \|\nabla\phi_\omega\|^2_{L^2} - \frac{b^2}{4}\|\rho\phi_\omega\|^2_{L^2} - \frac{3(p-1)}{2(p+1)}\|\phi_\omega\|^{p+1}_{L^{p+1}} = \partial_\lambda S_\omega(\phi^\lambda_\omega)|_{\lambda=1} = 0, \quad (1)$$

$$\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = \|\nabla\phi_\omega\|^2_{L^2} + \frac{3b^2}{4}\|\rho\phi_\omega\|^2_{L^2} - \frac{3(p-1)}{2(p+1)}\left\{\frac{3(p-1)}{2} - 1\right\}\|\phi_\omega\|^{p+1}_{L^{p+1}}. \quad (2)$$

To compare with our proof, we first recall the proof in Gonçalves Ribeiro [6], who proves $\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} < 0$ for the case where $1 + 4/3 + (4\sqrt{10} - 8)/9 \leq p < 5$ and $\omega > 0$.

Proof in Gonçalves Ribeiro [6]. ¿From (1) and (2), we have

$$\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = -\frac{3(p-1)-4}{2}\|\nabla\phi_\omega\|^2_{L^2} + \frac{3(p-1)+4}{2}\frac{b^2}{4}\|\rho\phi_\omega\|^2_{L^2}. \quad (3)$$

Moreover, from (1) and $I_\omega(\phi_\omega) = \langle S'_\omega(\phi_\omega), \phi_\omega\rangle = 0$, we have

$$\|\nabla\phi_\omega\|^2_{L^2} = \frac{5p-1}{5-p}\frac{b^2}{4}\|\rho\phi_\omega\|^2_{L^2} + \frac{3(p-1)}{5-p}\omega\|\phi_\omega\|^2_{L^2}. \quad (4)$$

¿From (3) and (4), we have

$$\partial^2_\lambda E(\phi^\lambda_\omega)|_{\lambda=1} = -\frac{9(p-1)^2 - 8(p+1)}{5-p}\frac{b^2}{4}\|\rho\phi_\omega\|^2_{L^2} - \frac{3(p-1)\{3(p-1)-4\}}{2(5-p)}\omega\|\phi_\omega\|^2_{L^2}.$$

Since we assume $\omega > 0$ and $1 + 4/3 < p < 5$, we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < -\frac{9(p-1)^2 - 8(p+1)}{5-p}\frac{b^2}{4}\|\rho\phi_\omega\|_{L^2}^2,$$

and $9(p-1)^2 - 8(p+1) \geq 0$ imples $p \geq 1 + 4/3 + (4\sqrt{10} - 8)/9$. □

Let us go back our proof. ¿From (1) and (2), we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = b^2\|\rho\phi_\omega\|_{L^2}^2 - \frac{9(p-1)}{4(p+1)}\left\{p - \left(1 + \frac{4}{3}\right)\right\}\|\phi_\omega\|_{L^{p+1}}^{p+1}.$$

Thus, we see that the condition $\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ is equivalent to

$$\frac{\|\rho\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} < \frac{9(p-1)}{4(p+1)b^2}\left\{p - \left(1 + \frac{4}{3}\right)\right\}. \tag{5}$$

We remark that the right hand side of (5) is a positive constant by the assumption $p > 1 + 4/3$ in Theorem 1. Therefore, to prove Thereom 1, it is enough to show

$$\lim_{\omega\to\infty} \frac{\|\rho\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} = 0.$$

To prove this, we rescale $\phi_\omega(x)$ as

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi_\omega}(\sqrt{\omega}x), \quad \omega \in (0,\infty).$$

Then, $\tilde{\phi_\omega}(x)$ satisfies

$$-\Delta\phi + \omega^{-2}\rho^2\phi + \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^3, \tag{6}$$

and we have

$$\frac{\|\rho\phi_\omega\|_{L^2}^2}{\|\phi_\omega\|_{L^{p+1}}^{p+1}} = \frac{\omega^{-2}\|\rho\tilde{\phi_\omega}\|_{L^2}^2}{\|\tilde{\phi_\omega}\|_{L^{p+1}}^{p+1}}.$$

Therefore, we have only to show

$$\lim_{\omega\to\infty} \frac{\omega^{-2}\|\rho\tilde{\phi_\omega}\|_{L^2}^2}{\|\tilde{\phi_\omega}\|_{L^{p+1}}^{p+1}} = 0. \tag{7}$$

Recall that $\psi_1(x)$ is the unique positive radial solution of (SP0) with $\omega = 1$ in $H^1(\mathbb{R}^n)$. We put

$$\tilde{I_\omega}(v) := \|\nabla v\|_{L^2}^2 + \omega^{-2}\|\rho v\|_{L^2}^2 + \|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1},$$
$$I_1^0(v) := \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2 - \|v\|_{L^{p+1}}^{p+1}.$$

The following is a key lemma to show (7). In fact, (7) follows from (i) and (iv) in Lemma 3.

Lemma 3. Let $1 < p < 5$ and $\phi_\omega(\rho, z)$ be the minimal action solution of (SP) in $H_{A,0}^1(\mathbb{R}^3)$ constructed in Proposition 1. Then, we have

$$\text{(i)} \quad \lim_{\omega\to\infty}\|\tilde{\phi_\omega}\|_{L^{p+1}}^{p+1} = \|\psi_1\|_{L^{p+1}}^{p+1}, \qquad \text{(ii)} \quad \lim_{\omega\to\infty} I_1^0(\tilde{\phi_\omega}) = 0,$$
$$\text{(iii)} \quad \lim_{\omega\to\infty}\|\tilde{\phi_\omega}\|_{H^1}^2 = \|\psi_1\|_{H^1}^2, \qquad \text{(iv)} \quad \lim_{\omega\to\infty}\omega^{-2}\|\rho\tilde{\phi_\omega}\|_{L^2}^2 = 0.$$

Remark. To prove (i) and (ii) in Lemma 3, we use the fact that $\tilde{\phi_\omega}(x)$ is a minimizer of

$$\inf\{\|v\|_{L^{p+1}}^{p+1} : v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\},\ \tilde{I}_\omega(v) \le 0\},$$

and $\psi_1(x)$ is a minimizer of

$$\inf\{\|v\|_{L^{p+1}}^{p+1} : v \in H^1(\mathbb{R}^n) \setminus \{0\},\ I_1^0(v) \le 0\}.$$

Parts (iii) and (iv) follow from (i) and (ii) immediately.

Remark. Under appropriate assumptions on $V(x)$, similar instability result to Theorem 1 can be proved for minimal action standing wave solution $e^{i\omega t}\phi_\omega(x)$ of

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \qquad (t,x) \in \mathbb{R}^{1+n},$$

when $p > 1 + 4/n$ and ω is sufficiently large (see [5]).

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