Global existence for a class of cubic nonlinear Schrödinger equations in one space dimension

Satoshi TONEGAWA (Tohoku University)

We consider the Cauchy problem for the nonlinear Schrödinger equation:

$$(1) \qquad \begin{cases} iu_t + \frac{1}{2}u_{xx} = F(u, \bar{u}, u_x, \bar{u}_x), & (t,x) \in \mathbf{R} \times \mathbf{R} \\ u(0,x) = u_0(x), & x \in \mathbf{R} \end{cases}$$

where $u = u(t,x)$ is a complex-valued function and F is a smooth function such that for some integer $p \geq 2$

$$F(u, \bar{u}, q, \bar{q}) = O(|u|^p + |q|^p)$$

near the origin. Our aim is to find nonlinearities F such that the Cauchy problem (1) has a unique global solution which is asymptotically free.

A lot of works established the local existence of solutions for the Cauchy problem (1) (see, for example, [6],[8]). In addition, when the degree p of a nonlinear term F is high (more precisely, when $p \geq 4$) results on the global existence of solutions and their asymptotic behavior were proved in the early 1980's([9],[13]). On the other hand, when the degree p is low($p = 2$ or 3), the global existence of solutions and their asymptotic behavior were studied for every particular nonlinearity F and results on them were proved for some nonlinearities ([2],[3],[7],[11],[12],etc.), but there were no results for the others.

As for cubic nonlinearities, most of works were long devoted to equations with nonlinearities satisfying the *gauge invariance* condition:

$$F(e^{i\theta}u, \overline{e^{i\theta}u}, e^{i\theta}q, \overline{e^{i\theta}q}) = e^{i\theta}F(u, \bar{u}, q, \bar{q}), \quad \theta \in \mathbf{R}$$

and few works to equations with the other nonlinearities. Recently, however, the global existence of solutions and their asymptotics to the Cauchy problem for nonlinear Schrödinger equations with cubic nonlinearities which may not be gauge invariant was established([4],[5],[10]).

Before stating our results, we introduce several notations here. For $1 \leq p \leq \infty$ and nonnegative integer m, we denote by L^p and $W^{m,p}$ the standard Lebesgue space and the Sobolev space, respectively. Let $H^m = W^{m,2}$. Let $C(I;B)$ and $C^1(I;B)$ denote the set of continuous functions and continuously differentiable functions from an interval I to a Banach space B, respectively. For a real a, let $[a]$ denote the largest integer not exceeding a. $\mathcal{F}f$ is the Fourier transform of f:

$$[\mathcal{F}f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} f(x) e^{-ix\xi} dx,$$

and $U(t) = e^{\frac{it}{2}\partial_x^2} = \mathcal{F}^{-1} e^{-\frac{it}{2}\xi^2} \mathcal{F}$ is the evolution operator associated with the free Schrödinger equation.

Here we state our results.

Theorem 1. Let m be an integer larger than or equal to 1 and let $F = cuu_x^2$, where c is a complex constant. We define a function $\varphi(u)$ by

$$\varphi(u) = \sum_{k=0}^{\infty} \frac{(-c)^k}{(2k+1)k!} u^{2k+1} \quad \left(= \int_0^u e^{-cz^2} dz \right).$$

Then, if $u_0 \in H^m$ and $||\mathcal{F}\varphi(u_0)||_{L^1}$ is sufficiently small, the Cauchy problem (1) has a unique global solution $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$. Moreover, the solution u is given explicitly by $u(t) = \varphi^{-1}\left(U(t)\varphi(u_0)\right)$. If in addition $u_0 \in L^1$, then there exists a unique $\phi \in H^m$ such that

$$||u(t)||_{L^\infty} = O(|t|^{-\frac{1}{2}}), \quad ||u(t) - U(t)\phi||_{H^m} = O(|t|^{-1}) \quad \text{as } t \to \pm\infty.$$

Furthermore, ϕ is given explicitly by $\phi = \varphi(u_0)$.

Theorem 2. Let m be an integer larger than or equal to 4 and let $F = c_1 \bar{u}\bar{u}_x^2 + c_2 \bar{u}_x^3$, where c_1 and c_2 are complex constants. Then, if $u_0 \in H^m \cap W^{[(m+5)/2],1}$ and $||u_0||_{H^m}, ||u_0||_{W^{[(m+5)/2],1}}$ is sufficiently small, the Cauchy problem (1) has a unique global solution $u \in C(\mathbf{R}; H^m) \cap C^1(\mathbf{R}; H^{m-2})$ and the solution u behaves like

$$||u(t)||_{H^m} = O(1), \quad ||u(t)||_{W^{[(m+1)/2],\infty}} = O(|t|^{-\frac{1}{2}}) \quad \text{as } t \to \pm\infty.$$

Moreover, there exist a unique $\phi_+ \in H^m$ and a unique $\phi_- \in H^m$ such that

$$||u(t) - U(t)\phi_+||_{H^m} = O(t^{-1}) \quad \text{as } t \to \infty,$$

$$||u(t) - U(t)\phi_-||_{H^m} = O(|t|^{-1}) \quad \text{as } t \to -\infty.$$

Remarks.

(1) If u is a solution of the Cauchy problem (1) with $F = cuu_x^2$, $\varphi(u)$ solves the free Schrödinger equation and satisfies $\varphi(u)|_{t=0} = \varphi(u_0)$, where φ is the function defined in Theorem 1. Since the free Schrödinger equation has a unique global solution $U(t)\varphi(u_0)$, our task is to find a condition which enables us restore the free solution $U(t)\varphi(u_0)$ to the solution of the original nonlinear Schrödinger equation. The assumption "$||\mathcal{F}\varphi(u_0)||_{L^1}$ is sufficiently small" given in Theorem 1 is a sufficient condition for it.

(2) The proof of Theorem 2 is based on the method of normal forms first introduced by Shatah[14] to study the quadratic nonlinear Klein-Gordon equation and on results on the boundedness of Fourier multipliers due to Coifman-Meyer[1].

(3) In his recent paper[10], Naumkin proved that the Cauchy problem for a large class of cubic nonlinear Schrödinger equations in one space dimension has a unique global solution and established the asymptotic formula of a solution in large time (Hayashi and Naumkin[5] gave a

simple proof and improved the result). The class of nonlinearities considered in [10] (and [5]) is so large that all the nonlinearities treated in Theorems 1 or 2 belong to it. Our results in Theorems 1 and 2, however, is not included in their results because of the difference of the assumptions on the initial data.

References

[1] R.Coifman and Y.Meyer, *Non-linear harmonic analysis, operator theory and P.D.E.*, Beijing Lectures in Harmonic Analysis, Princeton University Press (1986).

[2] N.Hayashi and P.I.Naumkin, *Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation*, Annales de l'Institut Henri Poincaré, **68** (1998), 159-177.

[3] N.Hayashi and P.I.Naumkin, *Asymptotics in large time of solutions to nonlinear Schrödinger and Hartree equations*, American Journal of Mathematics, **120** (1998), 369-389.

[4] N.Hayashi and P.I.Naumkin, *Large time behavior of solutions for derivative cubic nonlinear Schrödinger equations without a self-conjugate property*, Funkcialaj Ekvacioj, **42** (1999), 311-324.

[5] N.Hayashi and P.I.Naumkin, *Asymptotics of small solutions to nonlinear Schrödinger equations with cubic nonlinearies*, preprint.

[6] N.Hayashi and T.Ozawa, *Remarks on nonlinear Schrödinger equations in one space dimension*, Differential and Integral Equations, **7** (1994), 453-461.

[7] S.Katayama and Y.Tsutsumi, *Global Existence of Solutions for Nonlinear Schrödinger Equations in One Space Dimension*, Communications in Partial Differential Equations, **19** (1994), 1971-1997.

[8] C.Kenig, G.Ponce and L.Vega, *Small solutions to nonlinear Schrödinger equations*, Annales de l'Institut Henri Poincaré, **10** (1993), 255-288.

[9] S.Klainerman and G.Ponce, *Global, small amplitude solutions to nonlinear evolution equations*, Communications on Pure and Applied Mathematics, **36** (1983),133-141.

[10] P.I.Naumkin, *Cubic Derivative Nonlinear Schrödinger Equations*, SUT Journal of Mathematics, **36** (2000), No.1, 9-42.

[11] T.Ozawa, *Remarks on Quadratic Nonlinear Schrödinger Equations*, Funkcialaj Ekvacioj,**38** (1995), 217-232.

[12] T.Ozawa, *Finite Energy Solutions for the Schrödinger Equations with Quadratic Nonlinearity in One Space Dimension*, Funkcialaj Ekvacioj,**41** (1998), 451-468.

[13] J.Shatah, *Global existence of small solutions to nonlinear evolution equations*, Journal of Differential Equations, **46** (1982), 409-425.

[14] J.Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Communications on Pure and Applied Mathematics, **38** (1985), 685-696.