SELF-SIMILAR SOLUTIONS OF THE COUPLED SYSTEM OF SEMILINEAR WAVE EQUATIONS WITH DIFFERENT PROPAGATION SPEEDS

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In the present paper, we treat the coupled system of semilinear wave equations with different propagation speeds:

$$(\partial_t^2 - \Delta)\, u(x,t) = |u(x,t)|^{p_1} |v(x,t)|^{q_1}, \qquad x \in \mathbb{R}^3,\ t \geq 0, \tag{1}$$

$$(\partial_t^2 - s^2\Delta)\, v(x,t) = |u(x,t)|^{p_2} |v(x,t)|^{q_2}, \qquad x \in \mathbb{R}^3,\ t \geq 0, \tag{2}$$

$$u(x,0) = f_1(x), \qquad \partial_t u(x,0) = g_1(x), \qquad x \in \mathbb{R}^3, \tag{3}$$

$$v(x,0) = f_2(x), \qquad \partial_t v(x,0) = g_2(x), \qquad x \in \mathbb{R}^3. \tag{4}$$

where $s > 1$ or $0 < s < 1$, $p_1 + q_1 = p_2 + q_2 = \alpha + 1$. We call $(u(x,t), v(x,t))$ self-similar solution of (1)–(4) if the solution (u, v) of (1)–(4) satisfies $(u(x,t), v(x,t)) \equiv \lambda^{2/\alpha}(u(\lambda x, \lambda t), v(\lambda x, \lambda t))$ for any $\lambda > 0$. Our aim is to study the lower bound of p_j and q_j for the existence of self-similar solution. For that purpose, we prove the existence of global solutions including self-similar solutions. Before we proceed to our problem, we briefly recall the known results for the case of $s = 1$. Because those suggest what happens to our problem. For simplicity, we take the following single equation:

$$(\partial_t^2 - \Delta)\, u(x,t) = |u(x,t)|^{\alpha+1}, \qquad x \in \mathbb{R}^3,\ t \geq 0. \tag{5}$$

The existence of self-similar and related solutions of (5) is proved in [10] for $\alpha > 1 + \sqrt{5}$, in [7] for $\alpha > (\sqrt{13}+1)/3$ and in [8] and [3] for $\alpha > \sqrt{2}$. Pecher [8] also showed that even for small data no self-similar solutions in general exist if $\alpha \leq \sqrt{2}$. This optimal lower bound $\sqrt{2}$ already appeared in [4] on existence and blow-up for nonlinear wave equations of the type (5). If $\alpha \leq \sqrt{2}$, then there exist blow-up solutions.

On the other hand, Kubo [6] has very recently showed the existence of the global solution of (1)–(4) with $s \neq 1$ for $\alpha = \sqrt{2}$, $p_j \geq 1$, $q_j \geq 1$. He has not considered the case $\alpha < \sqrt{2}$. Moreover, his result has not implied the existence of self-similar solutions, because he has taken inhomogeneous data. Self-similarity of the solutions imposes homogeneity on the initial data, which causes additional difficulties. However, this result suggests that the discrepancy of propagation speeds may bring down the lower bound of α for the existence of self-similar solutions. Following Pecher's argument in [8], we prove the existence of global solutions for a class of data having a singularity at the origin and decaying slowly at ∞, which includes homogeneous data f_j and g_j of degree $-2/\alpha$ and $-2/\alpha - 1$, respectively, if $1 < \alpha < 2$, $p_j \geq 1$ and $q_j \geq 1$. These data are not necessarily radially symmetric. This result yields the existence of self-similar solution. Moreover, we show the existence of

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asymptotically self-similar solutions. We finally show that even for small data no self-similar solutions in general exist if $(p_1, q_1) \in \Omega$ or $(p_2, q_2) \in \Omega$, where

$$\Omega = \{(p,q) \big| 2/\alpha - 1 \geq \min\{p^{-1}, q^{-1}\}\}.$$

This result implies that the condition $\alpha > 1$ for the existence of self-similar solutions is sharp in the case $p_1 = q_1 = p_2 = q_2$.

Related results for the Schrödinger and the heat equations are proved by Cazenave-Weissler [1],[2] and Ribaud-Youssfi [9].

Remark 1. The nonlinear scattering theory has been useful for the study of asymptotic behavior in large time of solution to the nonlinear wave equation. On the other hand, the analysis of self-similar solutions and their stability may produce a different type of approach to this problem. For that purpose, the classification and the characterization of self-similar solutions would be important.

Before we mention main theorems, we give some notations. We define the operators as follows:

$$K_s(f,g)(x,t) = \frac{1}{4\pi s^2 t} \int_{|x-y|=st} g(y) dS_y + \partial_t \Big(\frac{1}{4\pi s^2 t} \int_{|x-y|=st} f(y) dS_y \Big),$$

$$L_s(F)(x,t) = \frac{1}{4\pi s^2} \int_0^t \frac{1}{t-t'} \Big(\int_{|x-y|=s(t-t')} F(y,t') dS_y \Big) dt'.$$

We define the space $V_{1,s}$ as follows:

$$V_{1,s} = \{(u,v) | u(x,t) \in C^0(\mathbb{R}^3 \times \mathbb{R}^+ \setminus \{|x| = t\}),$$
$$v(x,t) \in C^0(\mathbb{R}^3 \times \mathbb{R}^+ \setminus \{|x| = st\}), \|(u,v)\|_{V_{1,s}} < \infty\},$$

where

$$\|(u,v)\|_{V_{1,s}} = \|u\|_{V_1} + \|v\|_{V_s},$$
$$\|u\|_{V_s} = \sup_{|x| \neq st} (|x| + t)||x| - st|^{2/\alpha - 1} |u(x,t)|.$$

We put $\mathbb{R}^3_* = \mathbb{R}^3 \setminus \{0\}$.

We first mention the existence of global solutions to the Cauchy problem (1)–(4) including self-similar solutions. More precisely, we consider the corresponding integral equation:

$$u = K_1(f_1, g_1) + L_1(|u|^{p_1} |v|^{q_1}), \tag{6}$$

$$v = K_s(f_2, g_2) + L_s(|u|^{p_2} |v|^{q_2}). \tag{7}$$

Theorem 1. *Let $2 > \alpha > 1$ and $p_j \geq 1$, $q_j \geq 1$ for $j = 1,2$. Assume $f_j(x) \in C^1(\mathbb{R}^3_*)$, $g_j(x) \in C^0(\mathbb{R}^3_*)$ satisfy*

$$|f_j(x)| \leq \epsilon |x|^{-2/\alpha}, \quad |g_j(x)| + |\nabla f_j(x)| \leq \epsilon |x|^{-2/\alpha - 1},$$

for $j = 1,2$. If $\epsilon > 0$ is sufficiently small, then there exists a unique global solution (u,v) of (6)–(7) in $V_{1,s}$. Furthermore (u,v) satisfies $\|(u,v)\|_{V_{1,s}} \leq C\epsilon$, where C is a positive constant independent of ϵ.

Corollary 1. *Let*

$$f_j(x) = \epsilon F_j(x/|x|) |x|^{-2/\alpha}, \quad g_j(x) = \epsilon G_j(x/|x|) |x|^{-2/\alpha - 1},$$

where $F_j \in C^1$, $G_j \in C^0$ for j=1,2. If all the other assumptions of Theorem 1 hold, then the integral equation (6)–(7) has a unique global self-similar solution (u,v) in $V_{1,s}$. Furthermore (u,v) satisfies $\|(u,v)\|_{V_{1,s}} \le C\epsilon$, where C is a positive constant independent of ϵ.

We next mention the existence of a class of solutions which asymptotically behave like self-similar solutions. We define the cut functions $\chi \in C^1(\mathbb{R}^3_*)$ and $\widetilde{\chi} \in C^0(\mathbb{R}^3_*)$ as follows:

$$\chi(x), \widetilde{\chi}(x) = \begin{cases} 1 & \text{for } |x| \ge 1, \\ 0 & \text{for } |x| < 1/2. \end{cases}$$

Theorem 2. *Assume $2 > \alpha > 1$ and $p_j \ge 1$, $q_j \ge 1$ for $j = 1,2$ and $\delta \ge 0$. Let*

$$f_j(x) = \epsilon F_j(x/|x|)|x|^{-2/\alpha}, \qquad g_j(x) = \epsilon G_j(x/|x|)|x|^{-2/\alpha-1},$$

with $F_j \in C^1$ and $G_j \in C^0$ for $j = 1,2$. Let $f_j' = \chi f_j + \phi_j$, $g_j' = \widetilde{\chi} g_j + \psi_j$, where $\phi_j \in C^1(\mathbb{R}^3_*)$, $\psi_j \in C^0(\mathbb{R}^3_*)$ satisfy

$$|\phi_j| \le \begin{cases} \epsilon|x|^{-2/\alpha-\delta} & \text{for } |x| \ge 1, \\ \epsilon|x|^{-2/\alpha} & \text{for } |x| < 1, \end{cases} \quad |\psi_j| + |\nabla\phi_j| \le \begin{cases} \epsilon|x|^{-2/\alpha-\delta-1} & \text{for } |x| \ge 1, \\ \epsilon|x|^{-2/\alpha-1} & \text{for } |x| < 1, \end{cases}$$

for $j = 1,2$. If $\epsilon > 0$ is sufficiently small, then there exists a unique solution (u',v') of (6)–(7) with initial data (f_j', g_j'), which satisfies

$$\sup_{|x|\ne t} (|x|+t)||x|-t|^{2/\alpha-1}|u'(x,t)| \le C\epsilon,$$

$$\sup_{|x|\ne st} (|x|+t)||x|-st|^{2/\alpha-1}|v'(x,t)| \le C\epsilon.$$

If (u,v) denote the self-similar solution of (6)–(7) with initial data (f_j, g_j) (see Corollary 1) and if $\delta \ge 0$ satisfies

$$0 \le \delta < 1 - (2/\alpha - 1)\max\{p_1, q_1, p_2, q_2\},$$

then the following estimates hold:

$$|u(x,t) - u'(x,t)| \le C\epsilon(|x|+t)^{-1}||x|-t|^{-2/\alpha+1}(1+||x|-t|)^{-\delta}, \tag{8}$$

$$|v(x,t) - v'(x,t)| \le C\epsilon(|x|+t)^{-1}||x|-st|^{-2/\alpha+1}(1+||x|-st|)^{-\delta}. \tag{9}$$

We finally mention non-existence of self-similar solutions. We consider the Cauchy problem:

$$\begin{aligned} &(\partial_t^2 - \Delta)\, u(x,t) = |u(x,t)|^{p_1}|v(x,t)|^{q_1}, \\ &(\partial_t^2 - s^2\Delta)\, v(x,t) = |u(x,t)|^{p_2}|v(x,t)|^{q_2}, \\ &u(x,0) = 0, \qquad \partial_t u(x,0) = \epsilon|x|^{-2/\alpha-1}, \\ &v(x,0) = 0, \qquad \partial_t v(x,0) = \epsilon|x|^{-2/\alpha-1}, \end{aligned}$$

via its corresponding integral equation:

$$u = K_1(0, \epsilon|x|^{-2/\alpha-1}) + L_1(|u|^{p_1}|u|^{q_1}), \tag{10}$$

$$v = K_s(0, \epsilon|x|^{-2/\alpha-1}) + L_s(|v|^{p_2}|v|^{q_2}). \tag{11}$$

Theorem 3. *Let $(p_1,q_1) \in \Omega$ or $(p_2,q_2) \in \Omega$. Then, for any $\epsilon > 0$, there exist no solutions of (10)–(11), which take finite values for a.e. t and x.*

Theorem 3 shows that even for small data no time local self-similar solutions in general exist if $(p_1, q_1) \in \Omega$ or $(p_2, q_2) \in \Omega$. Here, we mention preliminary estimates.

Proposition 4. *If $u \in C^1(\mathbb{R}^3_*)$, $v \in C^0(\mathbb{R}^3_*)$ and*

$$|f(x)| \le \epsilon |x|^{-k}, \quad |g(x)| + |\nabla f(x)| \le \epsilon |x|^{-k-1},$$

for some $\epsilon > 0$ and $k > 1$, then the following estimate holds for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$:

$$|K_s(f,g)(x,t)| \le C\epsilon (|x| + t)^{-1} ||x| - st|^{-k+1}.$$

This proposition is the estimate for the linear wave equation. For the proof of Proposition 4, see, e.g., [8].

Proposition 5. *Assume $0 < s < 1$ or $1 < s$ and $\mu, \mu_1, \mu_2 > 0$. Let*

$$F(y,t') = (|y| + t')^{-1-\mu-\mu_1-\mu_2} ||y| - t'|^{\mu_1 - 1} |y| - st'|^{\mu_2 - 1}.$$

Then, the following estimates hold for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$:

$$|L_1(F)(x,t)| \le C(|x| + t)^{-1} ||x| - t|^{-\mu}, \tag{12}$$

$$|L_s(F)(x,t)| \le C(|x| + t)^{-1} ||x| - st|^{-\mu}. \tag{13}$$

This proposition is the estimate for the nonlinear terms. H. Kubo [6] proved the inhomogeneous version of this estimate. Following his argument, we can prove Proposition 5. Now, we only prove Theorem 1.

Proof of Theorem 1. We define the set X as follows:

$$X = \{(u,v) \in V_{1,s} \,|\, \|(u,v)\|_{V_{1,s}} \le C\epsilon\}.$$

We have only to prove that a map M defined by

$$M(u,v) = (K_1(f_0, f_1), K_s(g_0, g_1)) + (L_1(|u|^{p_1}|v|^{q_1}), L_s(|u|^{p_2}|v|^{q_2})),$$

is a contraction in X. ¿From Proposition 4, we have

$$\|(K_1(f_1, g_1), K_s(f_2, g_2))\|_{V_{1,s}} \le C\epsilon. \tag{14}$$

Let $h(y,t') = (|y| + t')^{-(p_1+q_1)} ||y| - t'|^{-(2/\alpha - 1)p_1} ||y| - st'|^{-(2/\alpha - 1)q_1}$. Then, we have

$$|L_1(|u|^{p_1}|v|^{q_1})(x,t)| \le C|L_1(h)(x,t)| \|u\|_{V_1}^{p_1} \|v\|_{V_s}^{q_1}. \tag{15}$$

Proposition 5 with $\mu = 2/\alpha - 1 > 0$, $\mu_1 = -(2/\alpha - 1)p_1 + 1 > 0$, $\mu_2 = -(2/\alpha - 1)q_1 + 1 > 0$ leads to

$$|L_1(h)(x,t)| \le C(|x| + t)^{-1} ||x| - t|^{-2/\alpha + 1}. \tag{16}$$

Therefore, we have by (15) and (16)

$$\|L_1(|u|^{p_1}|v|^{q_1})\|_{V_1} \le C\|(u,v)\|_{V_{1,s}}^{\alpha+1}. \tag{17}$$

In the same manner, we have

$$\|L_s(|u|^{p_2}|v|^{q_2})\|_{V_s} \le C\|(u,v)\|_{V_{1,s}}^{\alpha+1}. \tag{18}$$

Because

$$\begin{aligned} ||u|^{p_1}|v|^{q_1} - |u'|^{p_1}|v'|^{q_1}| \le & ||u|^{p_1}|v|^{q_1} - |u|^{p_1}|v'|^{q_1}| \\ & + ||u|^{p_1}|v'|^{q_1} - |u'|^{p_1}|v'|^{q_1}| \\ \le & C(|u|^{p_1}|v|^{q_1 - 1}|v - v'| + |u|^{p_1}|v'|^{q_1 - 1}|v - v'| \\ & + |u|^{p_1 - 1}|v'|^{q_1}|u - u'| + |u'|^{p_1 - 1}|v'|^{q_1}|u - u'|), \end{aligned}$$

we have

$$
\begin{aligned}
(19)\qquad & |L_1(|u|^{p_1}|v|^{q_1}) - L_1(|u'|^{p_1}|v'|^{q_1})| \\
\le & C\{|L_1(|u|^{p_1}|v|^{q_1-1}|v-v'|)| + |L_1(|u|^{p_1}|v'|^{q_1-1}|v-v'|)| \\
& + |L_1(|u|^{p_1-1}|v'|^{q_1}|u-u'|)| + |L_1(|u'|^{p_1-1}|v'|^{q_1}|u-u'|)|\} \\
\le & C|L_1(h)|\{\|u\|_{V_1}^{p_1}\|v\|_{V_s}^{q_1-1}\|v-v'\|_{V_s} + \|u\|_{V_1}^{p_1}\|v'\|_{V_s}^{q_1-1}\|v-v'\|_{V_s} \\
& + \|u\|_{V_1}^{p_1-1}\|v'\|_{V_s}^{q_1}\|u-u'\|_{V_1} + \|u'\|_{V_1}^{p_1-1}\|v'\|_{V_s}^{q_1}\|u-u'\|_{V_1}\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
(20)\qquad & \|L_1(|u|^{p_1}|v|^{q_1}) - L_1(|u'|^{p_1}|v'|^{q_1})\|_{V_1} \\
\le & C(\|(u,v)\|_{V_{1,s}}^{\alpha} + \|(u',v')\|_{V_{1,s}}^{\alpha})\|(u-u',v-v')\|_{V_{1,s}},
\end{aligned}
$$

from (16). In the same manner, we have

$$
\begin{aligned}
(21)\qquad & \|L_s(|u|^{p_2}|v|^{q_2}) - L_s(|u'|^{p_2}|v'|^{q_2})\|_{V_s} \\
\le & C(\|(u,v)\|_{V_{1,s}}^{\alpha} + \|(u',v')\|_{V_{1,s}}^{\alpha})\|(u-u',v-v')\|_{V_{1,s}},
\end{aligned}
$$

By collecting (14),(17) and (18), we obtain

$$
\|M(u,v)\|_{V_{1,s}} \le C\epsilon + C\|(u,v)\|_{V_{1,s}}^{\alpha+1}.
$$

By collecting (20)–(21), we also obtain

$$
\|M(u,v) - M(u',v')\|_{V_{1,s}} \le C(\|(u,v)\|_{V_{1,s}}^{\alpha} + \|(u',v')\|_{V_{1,s}}^{\alpha})\|(u-u',v-v')\|_{V_{1,s}}.
$$

The rest of the proof is standard. □

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References

[1] T. Cazenave and F. B. Weissler, *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations,* Math. Z. **228** (1998), 83–120.

[2] T. Cazenave and F. B. Weissler, *More self-similar solutions of the nonlinear Schrödinger equation,* Nonlinear Diff. Equ. and Applications **5** (1998), 355–365.

[3] K. Hidano, *Scattering and self-similar solutions for the nonlinear wave equation,* preprint.

[4] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions,* Manuscripta math. **28** (1979), 235–268.

[5] O. Kavian and F. B. Weissler, *Finite energy self-similar solutions of a nonlinear wave equation,* Comm. PDE **15** (1990), 1381–1420.

[6] H. Kubo, *Critical global existence of mild solutions to systems of semilinear wave equations,* preprint.

[7] H. Pecher, *Self-similar and asymptotically self-similar solutions of nonlinear wave equations,* Math. Ann. **316** (2000), 259–281.

[8] H. Pecher, *Sharp existence results for self-similar solutions of semilinear wave equations,* to appear in Nonlinear Diff. Equ. and Applications.

[9] F. Ribaud and A. Youssfi, *Regular and self-similar solutions of nonlinear Schrödinger equations,* J. Math. Pures Appl. **77** (1998), 1065–1079.

[10] F. Ribaud and A. Youssfi, *Solutions globales et solutions auto-similaires de l'équation des ondes non linéaire,* C. R. Acad. Sci. Paris, Série 1 **329** (1999), 33–36.

[11] J. Schaeffer, *The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p,* Proc. Roy. Soc. Edinburgh **101A** (1985), 31–44.