Blow-up problem to nonlinear heat equation with a nonlocal convection term *

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Let us consider blow-up problem to the following nonlinear nonlocal parabolic equation:

$$u_t = u_{xx} + \mu u_x \int_0^x u^{p-1} d\tilde{x} + u^p, \quad x \in \Omega,\ t > 0. \tag{1}$$

Here μ and $p > 1$ are real parameters and the domain Ω is whole interval $\mathbb{R}$ or a bounded interval $(-a, a), (a > 0)$. When Ω is $\mathbb{R}$ we consider the Cauchy problem (it is abbreviated to (CP)):

$$\begin{cases} u(x,0) &= u_0(x) > 0, & x \in \Omega, \\ \lim\limits_{|x|\to\infty} u(x,t) &= 0, & t > 0. \end{cases}$$

When Ω is bounded we consider the Dirichlet problem (it is abbreviated to (DP)):

$$\begin{cases} u(x,0) &= u_0(x) > 0, & x \in \Omega, \\ u(x,t) &= 0, & x \in \partial\Omega,\ t > 0. \end{cases}$$

In the case where $\mu = 0$ the properties of the blow-up solutions to the problem, which is called problem of Fujita type, are extensively studied by numerous authors. Among of these clarified properties, one of most interesting feature is that there exists a blow-up solution of Type II providing the space dimension and the exponent p is large enough ([4]) but all positive blow-up solutions are of Type I providing p is Sobolev subcritical. Let us introduce the notion of blow-up of Type I and II.

*Workshop on nonlinear dispersive and wave equations, Nov. 2000, at Science University of Tokyo

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Definition 1 *If the solution blows up in the same rate as the blow-up rate of the solution of ODE $\dot{u} = u^p$, we call it blow-up solution of* **Type I**. *On the other hand, if the solution blows up with faster rate than the solution of Type I, we call it blow-up solution of* **Type II**.

What we would like to understand is what kind of structure of the equation can allow to possess the blow-up solutions of Type II. Hence here we concern the effect of the nonlocal convection term on the occurrence of blow-up and the rate of blow-up.

We would like to note that there are several problems which can be described by our equation with nonzero μ. Such problems arise in plasma physics (see [6]), fluid dynamics (see [2]), and biological mathematics (see [5]).

Our main theoretical result is the following.

Theorem 1

(A) *For the problem* (CP), *if* $0 \leq \mu < 1$ *then there exists blow-up solution. Moreover, if* $0 \leq \mu < 1$ *and* $p < 3$ *then all solutions blow up in finite time.*

(B) *For the problem* (DP), *if* $0 \leq \mu < 1$ *then there exists blow-up solution, if* $p - 1 \leq \mu$ $(p \geq 2)$ *then all non-negative solutions exist globally.*

(C) *For the problem* (CP), *if* $\mu = 1$ *then there exist blow-up solutions.*

(D) *For the problem* (CP), *if* $\mu = 1$ *then there exists a family of positive steady state solutions and if* $0 \leq \mu < 1$ *there is no nonnegative, non-trivial steady state solution.*

Unfortunately we have not obtained any result on the blow-up rate yet. Hence we resorted to numerical method. Here we employed so-called rescaling algorithm, which was invented by Berger and Kohn [1], as numerical method. This method is based on the scaling invariance of the equations. Our equation has the desired scaling invariance, namely if u is a solution of the equation 1 then $u_\lambda(y,\tau) = \lambda^{\frac{2}{p-1}} u(\lambda y, \lambda^2 \tau)$ is also a solution for any $\lambda > 0$. Hence we can adopt this method. Because of this use of the scaling invariance, we can know that the blow-up rate belongs which types (see [1]).

From our numerical simulation, we observed the following.

Conjecture 1 $0 \leq \mu < 1 \Rightarrow$ *blow-up solution is of Type I.*
$\mu = 1, p < 3 \Rightarrow$ *there is no blow-up solution.*
$\mu = 1, p = 3 \Rightarrow$ *blow-up solution is of Type II.*
$\mu = 1, p > 3 \Rightarrow$ *blow-up solution is of Type I.*
$\mu > 1 \Rightarrow$ *there is no blow-up solution.*

Based on the theoretical and numerical results above, we conclude the following:

> A kind of convection terms might cause the blow-up of Type II. In the case of equation (1), Type II blow-up is quite rare phenomenon.

References

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