LARGE TIME BEHAVIOR OF SMALL SOLUTIONS TO SUBCRITICAL DERIVATIVE NONLINEAR SCHRÖDINGER EQUATIONS

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§1. Introduction

This is a joint work with Nakao Hayashi[1] and Parvel I.Naumkin[2][16].

We consider the Cauchy problem for the cubic derivative nonlinear Schrödinger equation

$$\begin{cases} iu_t + \frac{1}{2}u_{xx} = a(t)F(u,u_x), & (t,x) \in \mathbf{R}^2, \\ u(0,x) = \epsilon u_0, & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where the coefficient $a(t)$ satisfies the following time growth condition $|a(t)| \leq C(1+|t|)^{1-\delta}, 0 < \delta < 1, \epsilon$ is a sufficiently small constant and the nonlinear interaction term F consists of the cubic nonlinearities of derivative type

$$\begin{aligned} F(u,u_x) = {} & \lambda_1|u|^2u + i\lambda_2|u|^2u_x + i\lambda_3u^2\overline{u}_x + \lambda_4|u_x|^2u \\ & + \lambda_5\overline{u}u_x^2 + i\lambda_6|u_x|^2u_x, \end{aligned} \tag{1.2}$$

where the coefficients $\lambda_1, \lambda_6 \in \mathbf{R}, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbf{C}, \lambda_2 - \lambda_3 \in \mathbf{R}, \lambda_4 - \lambda_5 \in \mathbf{R}$. We propose to classify the nonlinearities in a nonlinear Schrödinger equation as follows. If the usual scattering states exist in $\mathbf{L}^2$, we call the nonlinearity as a supercritical one. If the usual scattering states do not exist and the $\mathbf{L}^2$ norm of the nonlinear term decays like $O(t^{-\delta})$ we call it as a critical one, if $\delta = 1$ and as a subcritical one,if $\delta < 1$. Our main result in this present work says that nonlinearity (1.2) is the supercritical one if $b(\xi) = 0$ and $1/2 < \delta \leq 1$, where

$$b(\xi) = \lambda_1 - (\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 - \lambda_6\xi^3, \tag{1.3}$$

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it is critical, if $b(\xi) \neq 0$ and $\delta = 1$, and it is subcritical, if $b(\xi) \neq 0$ and $\delta \in (0,1)$. In the case of power nonlinearity $\lambda|u|^{p-1}u$ if $\lambda > 0$, and it appears to be a supercritical one for $p > 3$, a critical one for $p = 3$, and a subcritical one for $1 < p < 3$. Non-local nonlinear Schrödinger equation with nonlinearity $\lambda \int |x-y|^{-\delta}|u|^2 dyu$, or $\lambda|t|^{1-\delta}|u|^2 u, \lambda \in \mathbf{R}, \delta \in (0,1)$ is also classified as a subcritical one (see paper[12]). Note that cubic nonlinearity is not necessarily critical, for example,the nonlinearity $\partial_x(u^3)$ is supercritical due to it's oscillating structure (see [13]).Also note that the quadratic nonlinearity $(u_x)^2$ is supercritical (see [23]) and we do not know, when a quadratic nonlinearity is critical or subcritical.There are many works concerning the nonlinear Schrödinger equations with power nonlinearities . In the supercritical case the scattering problem was studied in papers [3], [4], [8], [18], [20], [24], [25], [26] and in the critical case it was treated in [2], [11], [15], [21], [22]. The existence of modified scattering states was shown in papers [11], [15], [21] and modified wave operators were constructed in [2], [22]. In [15], they considered the Cauchy problem (1.1) with cubic derivative nonlinearity (1.2), when the coefficient $\delta = 1$. It is proved that the solution of (1.1) with (1.2) and $\delta = 1$ exists and satisfies the sharp time decay estimate $\|u(t)\|_{\mathbf{L}^p} \leq C\epsilon t^{1/p-1/2}$ for all $t \geq 1$, where $2 \leq p \leq \infty$, if the initial fuction u_0 is in the weighted Sobolev space $\mathbf{H}^{3,0} \cap \mathbf{H}^{2,1}$ defined below in Notation and function spaces. Moreover it is shown that there exists a unique final state $u_+ \in \mathbf{L}^2$ such that

$$\left\| u(t) - \exp\left(-i|\hat{u}_+(\cdot/t)|^2 b(\cdot/t)\log t\right)\mathcal{U}(t)u_+ \right\|_{\mathbf{L}^2} = O\left(t^{-1/2-\alpha}\right)$$

for all $t \geq 1$, where $0 < \alpha < 1/4$, $\hat{\phi}$ denotes the Fourier transform of ϕ, $\mathcal{U}(t)$ is the free Schrödinger evolution group.The aim of the present work is to prove the results of paper [15] for the more difficult subcritical cases. As far as we know there are no results for the scattering problem in subcritical cases except the case of the nonlinear Schrödinger equation without derivatives of unknown function in the nonlinear term (see [5], [6], [7], [9], [10], [12]). The main point in the proof of our results is the applicaton of the transformation for the solution $u(t) = \mathcal{M}(t)\mathcal{D}(t)v(t)$ (see [17]), where the operators $\mathcal{M}(t)$ and $\mathcal{D}(t)$ are defined below in Notation and function spaces. The idea of the transformation can be understood from the large time asymptotic expansion of the solution of the linear Schrödinger equation. But the miracle property of the transformation is that the nonlinearity also can be easily transformed and in the resulting equation for the new unknown function $v(t)$ we can extract the divergent nonlinear term explicitly. To cancel these divergent terms we present the solution $v(t)$ in the rapidly oscillating form $v = we^{ig}$ with slowly varying amplitude $w(t)$ and growing with time phase function $g(t)$. However the system of equations for w and g has the so-called difficulty of the derivative loss. In order to prove the existence of the solution (w, g) we apply the estimates in the analytic function space $\mathcal{A}^{5/2,3}_{\sigma(t)}$, where $\sigma(t) = \beta(1 + (1+t)^{-\gamma})/2, \gamma \in (0, \delta/2]$. Analytic function spaces was used by many authors to overcome the derivative loss in the nonlinear evolution equations, e.g., see([1], [9], [10], [19]). In what follows we consider the positive time t only since for the negative one the results are analogous.

To state our results precisely,we now give

Notation and Fuction spaces: Let $\mathcal{F}\phi$ or $\hat{\phi}$ be the Fourier transform of ϕ defined by $\mathcal{F}\phi(\xi) = (2\pi)^{-1/2}\int e^{-ix\xi}\phi(x)dx$ and $\mathcal{F}^{-1}\phi(x)$ be the inverse Fourier transform of ϕ, i.e. $\mathcal{F}^{-1}\phi(x) = (2\pi)^{-1/2}\int e^{ix\xi}\phi(\xi)d\xi$. We introcuce some function spaces. Weighted Sobolev space is $\mathbf{H}^{m,s} = \{\varphi \in \mathbf{L}^2; \|\varphi\|_{m,s} < \infty\ , m,s \in \mathbf{R}\}$, where $\|\varphi\|_{m,s} = \|\langle x\rangle^s\langle i\partial_x\rangle^m\varphi\|_{\mathbf{L}^2}\ , \langle x\rangle = \sqrt{1+x^2}$. Also we define the analytic function space $\mathcal{A}^{m,s}_\sigma = \{\varphi \in \mathbf{L}^2; \|\varphi\|_{\mathcal{A}^{m,s}_\sigma} < \infty\ , m,s \in \mathbf{R}\}$, $\sigma = \sigma(t) = \frac{\beta}{2}\left(1+(1+t)^{-\gamma}\right), \gamma \in \left(0, \frac{\delta}{2}\right]$ with norm $\|\varphi\|_{\mathcal{A}^{m,s}_\sigma} = \|e^{\sigma|p|}\langle p\rangle^m\langle i\partial_p\rangle^s\hat{\varphi}(p)\|_{\mathbf{L}^2}$, which can be expressed in $x-$representation in terms of the analyticity in the strip $-\sigma \le Imz \le \sigma$ via the following norm $\|\varphi(\cdot + i\sigma)\|_{m,s} + \|\varphi(\cdot - i\sigma)\|_{m,s}$. Indeed we have the inequalities $\frac{1}{C}\|\varphi\|_{\mathcal{A}^{m,s}_\sigma} \le \|\varphi(\cdot + i\sigma)\|_{m,s} + \|\varphi(\cdot - i\sigma)\|_{m,s} \le C\|\varphi\|_{\mathcal{A}^{m,s}_\sigma}$ with $C > 0$. The free Schrödinger evolution group $\mathcal{U}(t)$ gives us the solution of the linear Cauchy problem (1.1) (with $F = 0$). It can be represented explixitly in the following manner.

$$\mathcal{U}(t)\varphi = (2\pi it)^{-1/2}\int e^{i(x-y)^2/2t}\varphi dy = \mathcal{F}^{-1}e^{-it\xi^2/2}\mathcal{F}\varphi.$$

Note that $\mathcal{U}(t) = \mathcal{M}(t)\mathcal{D}(t)\mathcal{F}\mathcal{M}(t)$, where $\mathcal{M}(t) = \exp(ix^2/2t)$ and $\mathcal{D}(t)$ is the dilation operator defined by $(\mathcal{D}(t)\psi)(x) = \frac{1}{\sqrt{it}}\psi\left(\frac{x}{t}\right)$. Then since $\mathcal{D}^{-1}(t) = i\mathcal{D}(1/t)$ we have $\mathcal{U}(-t) = \overline{\mathcal{M}}\mathcal{F}^{-1}\mathcal{D}^{-1}(t)\overline{\mathcal{M}} = i\overline{\mathcal{M}}\mathcal{F}^{-1}\mathcal{D}(1/t)\overline{\mathcal{M}}$,where $\overline{\mathcal{M}} = \mathcal{M}(-t)$. Let $\mathbf{C}(\mathbf{I};\mathbf{B})$ be the space of continuous functions from a time interval $\mathbf{I}$ to a Banach space $\mathbf{B}$. Different positive constants we denote everywhere by the same letter C. We use the following fuction spaces:

$$\mathbf{B} = \{\psi \in \mathbf{C}([0,T];\mathbf{L}^2); \|\psi\|^2_{\mathbf{B}} = \sup_{0\le t\le T}\|\mathcal{U}(-t)\psi\|^2_{\mathcal{A}^{\frac{5}{2},3}_{\sigma(t)}} + 2\int_0^T |\sigma'(t)|\|\mathcal{U}(-t)\psi(t)\|^2_{\mathcal{A}^{3,3}_{\sigma(t)}}dt < \infty\}$$

$$\mathbf{X} = \left\{\begin{pmatrix}\psi\\ \phi\end{pmatrix}; \psi,\phi \in \mathbf{C}([1,\infty);\mathbf{L}^2), \left\|\begin{pmatrix}\psi\\ \phi\end{pmatrix}\right\|^2_{\mathbf{X}} < \infty\right\}$$

where

$$\left\|\begin{pmatrix}\psi\\ \phi\end{pmatrix}\right\|^2_{\mathbf{X}} = \sup_{1\le t<\infty}\left(\|\psi(t)\|^2_{\mathcal{A}^{\frac{5}{2},3}_{\sigma(t)}} + t^{2(\delta-1-\gamma)}\|\phi_\xi(t)\|^2_{\mathcal{A}^{\frac{5}{2},3}_{\sigma(t)}} + t^{2(\delta-1)}\|\phi(t)\|^2_{\mathcal{A}^{2,3}_{\beta/2}}\right)$$
$$+ \int_1^\infty |\sigma'(t)|\left(\|\psi(t)\|^2_{\mathcal{A}^{3,3}_{\sigma(t)}} + t^{2(\delta-1-\gamma)}\|\phi_\xi(t)\|^2_{\mathcal{A}^{3,3}_{\sigma(t)}}\right)dt, \qquad \gamma \in (0, \delta/2]$$

Now we state the result in this talk.

Theorem 1.1. *We assume that* $e^{\beta\langle x\rangle}u_0 \in \mathbf{H}^{3,\frac{5}{2}}, \beta > 0$. *Then there exists a unique global solution of the Cauchy problem for the cubic derivative nonlinear Schrödinger equation* (1.1) *such that* $u \in \mathbf{C}([0,\infty);\mathbf{H}^{3,0})$, $\mathcal{FU}(-t)u(t) \in \mathbf{C}([0,\infty);\mathcal{A}^{\frac{5}{2},3}_{\beta/2})$, *and the decay estimate*

$$\|u(t)\|_{\mathbf{L}^p} \leq C\epsilon t^{1/p-1/2}$$

is valid for all $t \geq 1$, *where* $2 \leq p \leq \infty$. *Moreover, if* $\frac{1}{2} < \delta < 1$, *there exists a unique final state* $\hat{u}_+ \in \mathcal{A}^{0,3}_{\beta/2}$ *such that the following asymptotics for* $t \to \infty$

$$u(t,x) = \frac{1}{\sqrt{it}}\hat{u}_+\left(\frac{x}{t}\right)\exp\left(\frac{ix^2}{2t} - i\int_0^t \frac{a(\tau)}{\tau+1}d\tau b\left(\frac{x}{t}\right)\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2\right) + O(t^{-1/2-\delta}) \tag{1.4}$$

is true uniformaly with respect to $x \in \mathbf{R}$, *where* $b(\xi) = \lambda_1 - (\lambda_2 - \lambda_3)\xi + (\lambda_4 - \lambda_5)\xi^2 - \lambda_6\xi^3$.

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