REMARKS ON THE COMPLEX GINZBURG-LANDAU AND NONLINEAR SCHRÖDINGER EQUATIONS

TOMOMI YOKOTA*

Department of Mathematics
Science University of Tokyo

Let Ω be a bounded domain in $\mathbf{R}^N$ $(N \geq 1)$ with C^2-boundary $\partial\Omega$. In this talk we consider the following initial-boundary value problem:

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\lambda \geq 0$, $\kappa \geq 0$, $\alpha, \beta, \gamma \in \mathbf{R}$ and the exponent $q > 1$ are constants; $i = \sqrt{-1}$, and u is a complex-valued unknown function. When $\lambda > 0$ and $\kappa > 0$, (P) is a problem for the usual complex Ginzburg-Landau equation (CGL). On the other hand, (P) is obviously reduced to a problem for the usual nonlinear Schrödinger equation (NLS) when $\lambda = \kappa = 0$. Both (CGL) and (NLS) have been intensively studied separately; nevertheless, there seems to be no preceding work which deals with both equations simultaneously. Recently Bechouche-Jüngel [**1**] constructed a weak solution of (NLS) by taking the inviscid limit of strong solutions of (CGL) as $\lambda \downarrow 0$ and $\kappa \downarrow 0$ when $\alpha\beta \geq 0$ (or $\alpha\beta < 0$ and $q \leq 2 + 4/N$).

The purpose of this talk is to prove the existence of weak and C^1-solutions to (P) including the case where $\lambda = 0$ or $\kappa = 0$. Before stating the result we define weak solutions to (P).

Definition. A function $u(\cdot) \in C([0,\infty); L^2(\Omega))$ is called a *weak solution* to (P) if

(i) $u(0) = u_0$ in $L^2(\Omega)$;

(ii) $u(\cdot) \in L^2(0,T; H_0^1(\Omega)) \cap L^q(0,T; L^q(\Omega))$ $\forall\, T > 0$;

(iii) For $T > 0$ and $\varphi(\cdot) \in C_0^\infty((0,T); H_0^1(\Omega) \cap L^q(\Omega))$,

$$-\int_0^T \Big(u(t), \frac{\partial\varphi}{\partial t}(t)\Big)_{L^2}\, dt + (\lambda + i\alpha)\int_0^T (\nabla u(t), \nabla\varphi(t))_{L^2}\, dt$$
$$+ (\kappa + i\beta)\int_0^T \langle |u(t)|^{q-2}u(t), \varphi(t)\rangle_{L^{q'}, L^q}\, dt - \gamma\int_0^T (u(t), \varphi(t))_{L^2}\, dt = 0,$$

where q' is the conjugate exponent to q: $1/q + 1/q' = 1$.

*Partially supported by JSPS Research Fellowships for Japanese Young Scientists (No. 00109).
E-mail: yokota@minserver.ma.kagu.sut.ac.jp

We present two kinds of existence theorems for weak solutions to (P).

Theorem 1. *Let $\lambda \geq 0, \kappa \geq 0, \alpha\beta \geq 0$ and $q > 1$. Then for any $u_0 \in H_0^1(\Omega) \cap L^q(\Omega)$ there exists a weak solution $u(\cdot)$ to* (P) *such that*

$$u(\cdot) \in L^\infty(0,T; H_0^1(\Omega) \cap L^q(\Omega)) \ \ \forall\, T > 0,$$
$$(\partial u/\partial t)(\cdot) \in L^\infty(0,T; H^{-1}(\Omega) + L^{q'}(\Omega)) \ \ \forall\, T > 0$$

with the estimates

$$\|u(t)\|_{L^2} \leq e^{\gamma t}\|u_0\|_{L^2} \ \ \forall\, t \geq 0,$$
$$\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{\delta}{q}\|u(t)\|_{L^q}^q \leq e^{M\gamma_+ t}\Big(\frac{1}{2}\|\nabla u_0\|_{L^2}^2 + \frac{\delta}{q}\|u_0\|_{L^q}^q\Big) \ \text{ a.a. } t \geq 0,$$

where $M := \max\{2, q\}$, $\gamma_+ := \max\{\gamma, 0\}$ and $\delta > 0$ is a constant depending only on $\lambda + i\alpha$, $\kappa + i\beta$ and q.

Theorem 2. *Let $\lambda \geq 0, \kappa > 0, \dfrac{|\beta|}{\kappa} \leq \dfrac{2\sqrt{q-1}}{|q-2|}$ and $q > 1$. Then for any $u_0 \in H_0^1(\Omega)$ there exists a unique weak solution $u(\cdot)$ to* (P) *such that*

$$u(\cdot) \in L^\infty(0,T; H_0^1(\Omega)) \cap L^q(0,T; L^q(\Omega)) \ \ \forall\, T > 0,$$
$$(\partial u/\partial t)(\cdot) \in L^\infty(0,T; H^{-1}(\Omega)) + L^{q'}(0,T; L^{q'}(\Omega)) \ \ \forall\, T > 0$$

with the estimates

$$\|u(t)\|_{H^1} \leq e^{\gamma t}\|u_0\|_{H^1} \ \ \forall\, t \geq 0,$$
$$\int_0^T \|u(t)\|_{L^q}^q\, dt \leq \frac{1}{2\kappa}\, e^{2\gamma_+ T}\|u_0\|_{L^2}^2 \ \ \forall\, T > 0.$$

Remark. 1) Theorem 2 is a slight improvement of Lions's result in [**3**, Chapitre 1, Section 10] in which it is assumed that $u_0 \in H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ and $\lambda = \beta = 0$.

2) If $\lambda > 0$ and $\kappa > 0$, then for any $u_0 \in L^2(\Omega)$ there exists a weak solution to (P) without any restriction on $\alpha, \beta \in \mathbf{R}$ and $q > 1$ (see Ginibre-Velo [**2**, Proposition 2.1]).

3) When $\lambda > 0$, $\kappa > 0$ and $\kappa^{-1}|\beta| \leq 2\sqrt{q-1}/|q-2|$, the existence of "strong" solutions to (P) with $u_0 \in L^2(\Omega)$ (smoothing effect) is proved by Okazawa-Yokota [**5**] (see also [**4**]).

Concerning the existence of C^1-solutions to (P) we first obtain the following local existence theorem by the same argument as in Ôtani [**6**, Theorem 8].

Theorem 3. *Let $N = 1, 2, 3$, $\lambda \geq 0$, $\kappa \geq 0$, $\alpha, \beta, \gamma \in \mathbf{R}$ and $q \geq 2$. Then for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists $T_m \in (0, \infty]$ such that* (P) *has a unique C^1-solution $u(\cdot) \in C^1([0, T_m); L^2(\Omega)) \cap C([0, T_m); H^2(\Omega) \cap H_0^1(\Omega))$. Furthermore, if $T_m < \infty$, then $\limsup_{t \uparrow T_m} \|u(t)\|_{L^\infty} = \infty$.*

Finally we present the global existence theorem for C^1-solutions to (P).

Theorem 4. *Let $\lambda \geq 0$, $\kappa \geq 0$, $\alpha, \beta, \gamma \in \mathbf{R}$ and $q \geq 2$. Assume that*

(i) $N = 1$; $\alpha\beta \geq 0$

or (ii) $N = 1, 2, 3$; $\kappa > 0$ *and* $\dfrac{|\beta|}{\kappa} < \dfrac{2\sqrt{q-1}}{q-2}$.

Then for any $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ there exists a unique global C^1-solution $u(\cdot) \in C^1([0,\infty); L^2(\Omega)) \cap C([0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$ to (P).

Remark. When $N = 3$ and $\lambda = \beta = 0$, Theorem 4 improves the existence theorem established by Pecher and von Wahl [**7**] in which it is assumed that $q \leq 6$.

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