# Global Existence of Classical Solutions for Nonlinear Wave Equations in Three Space Dimensions

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## 1. Introduction.

We consider the Cauchy problem

$$\partial_t^2 u_i - c_i^2 \Delta u_i = F_i(u, \partial u), \quad i = 1, \dots, m \qquad \text{for} \quad t > 0, \tag{1}$$

$$u_i = \varphi_i, \ \partial_t u_i = \psi_i, \quad i = 1, \dots, m \qquad \text{for} \quad t = 0, \tag{2}$$

where $u_i = u_i(t, x_1, x_2, x_3), u = (u_1, \dots, u_m)$ and $\partial u = (\partial_\alpha u_i)$. Here $\alpha$ ranges from 0 to 3 and $i$ from 1 to $m$. We consider the existence of global classical solutions to (1)-(2) when the nonlinearities in (1) are quadratic, especially the case where

$$F_i(u, \partial u) = \sum_{j,k=1}^{m} \sum_{\alpha=0}^{3} C_\alpha^{ijk} u_j \partial_\alpha u_k, \quad i = 1, \dots, m.$$

It is known that in three spatial dimensions, a solution of a wave equation with quadratic nonlinearities may develop a singularity in finite time, even for small initial data. For example, consider a single wave equation $\partial_t^2 u - \Delta u = \partial_t(u^2)$. If the initial data are given by $(u(0), \partial_t u(0)) = (0, \psi)$, $\psi \in C_0^\infty$ and $\psi \geq 0$, then the solution $u$ becomes singular in finite time, unless $u$ is identically zero. On the other hand, we also know a global existence result. Consider a system of wave equations

$$\begin{cases} \partial_t^2 u_1 - c_1^2 \Delta u_1 = \sum_\alpha A_\alpha \partial_\alpha \{(u_2)^2\}, \\ \partial_t^2 u_2 - c_2^2 \Delta u_2 = \sum_\alpha B_\alpha \partial_\alpha \{(u_1)^2\}, \end{cases}$$

and assume $c_1 \neq c_2$. If the initial data are in $C_0^\infty$ and sufficiently small, we have a unique global solution in $C^\infty$. So in order to find global solutions, we should think about the case $m \geq 2$.

In this article, we treat the case where the nonlinear terms $F_i(u, \partial u)$ are quadratic and do not have divergence forms. It is desirable that useful estimates of $\|u(t)\|_{L^2}$ are obtained, but the usual energy inequality do not estimate $u$ itself, which makes the analysis complicated. So we need rather strong assumptions to show the global existence.

## 2. Notation.

We define the vector fields

$$\Omega_1 = x_2\partial_3 - x_3\partial_2, \ \ \Omega_2 = x_3\partial_1 - x_1\partial_3, \ \ \Omega_3 = x_1\partial_2 - x_2\partial_1,$$

and set

$$\Gamma_\alpha = \partial_\alpha, \quad \alpha = 0, 1, 2, 3,$$

$$\Gamma_4 = \Omega_1, \ \ \Gamma_5 = \Omega_2, \ \ \Gamma_6 = \Omega_3.$$

We denote by $\Box_i$ the d'Alembertian $\partial_t^2 - c_i^2\Delta$. Then we have the commutation relations

$$\Gamma_\alpha \Box_i v = \Box_i \Gamma_\alpha v.$$

We consider the following quantities:

$$\langle u_i \rangle_{\rho,k,t} = \sum_{|a|\le k} \| \langle s + |y| \rangle \langle c_i s - |y| \rangle^{\rho} \Gamma^a u_i \|_{L^\infty(R_t)},$$

$$[\partial u_i]_{1+\rho,k,t} = \sum_{|a|\le k} \sum_{\alpha=0}^{3} \| \langle |y| \rangle \langle c_i s - |y| \rangle^{1+\rho} \Gamma^a \partial_\alpha u_i \|_{L^\infty(R_t)},$$

$$|u|_{\rho,k,t} = \sum_{i=1}^{m} (\langle u_i \rangle_{\rho,k,t} + [\partial u_i]_{1+\rho,k,t}),$$

where $R_t = \{ (s,y) \,|\, 0 \le s \le t, \ y \in \mathbf{R}^3 \}$,

$$\| \partial u(t) \|_k = \sum_{|a|\le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \| \Gamma^a \partial_\alpha u_i(t, \cdot) \|_{L^2},$$

$$\| \partial u \|_{k,t} = \sum_{|a|\le k} \sum_{i=1}^{m} \sum_{\alpha=0}^{3} \sup_{0\le s\le t} \| \Gamma^a \partial_\alpha u_i(s, \cdot) \|_{L^2}.$$

Then it follows

$$\| \partial u(t) \|_k \le C |u|_{\rho,k,t}.$$

## 3. Main Result.

We assume that any two of the propagation speeds $c_i$ are not the same. For the sake of clarity we number them as

$$0 < c_1 < \cdots < c_m.$$

Then the nonlinearities in (1) we consider are

$$F_i(u,\partial u)=\sum_{\alpha=0}^{3}\left(\sum_{j<k<i}C_\alpha^{ijk}u_j\partial_\alpha u_k+\sum_{i<k}C_\alpha^{iik}u_i\partial_\alpha u_k+\sum_{j<i}C_\alpha^{iji}u_j\partial_\alpha u_i\right),$$

where $C_\alpha^{ijk}$ are constants. We also assume that the initial data in (2) have compact supports. So there is a positive number $R$ such that

$$\varphi_i(x)=0,\ \ \psi_i(x)=0,\quad i=1,\dots,m,\qquad \text{for}\quad |x|\geq R.$$

We need these assumptions to assure

$$u_i(t,x)=0\qquad \text{for}\quad |x|\geq c_it+R,\ i=1,\dots,m,$$

which is important in our energy estimates.

**Theorem.** Let $\varphi_i\in H^{k+1},\psi_i\in H^k$ $(i=1,\dots,m)$ for some $k\geq 13$. There exists $\varepsilon>0$ such that for all $\{(\varphi_i,\psi_i)\}_{1\leq i\leq m}$ satisfying $(1+R)|u|_{2,13,0}\leq\varepsilon$, we have a unique solution $u\in\bigcap_{j=0}^{2}C^j([0,T];H^{k+1-j})$ of the Cauchy problem (1)-(2) for every $T>0$.

**Corollary.** Let $\varphi_i,\psi_i\in C_0^\infty$ $(i=1,\dots,m)$ and $(1+R)|u|_{2,13,0}\leq\varepsilon$. Then we have a unique solution $u\in C^\infty([0,T]\times\mathbf{R}^3)$ of the Cauchy problem (1)-(2) for every $T>0$.

To prove the theorem, we use a continuation principle in $\cap_j C^j([0,T];H^{s-j})$. It is sufficient to show that the first and the second derivatives of a solution $u$ on any interval $[0,T_0]$ are bounded independent of $T_0$. In fact, by assuming $(1+R)|u|_{2,13,0}$ to be small, we can prove that $|u|_{1,3,t}$ is bounded by a constant.

**References.**


[1] K. Hidano, K. Tsutaya, *Global existence and asymptotic behavior of solutions for nonlinear wave equations*, Indiana Univ. Math. J., **44**, 1995, 1273-1305.

[2] F. John, *Blow-up for quasi-linear wave equations in three space dimensions*, Comm. Pure Appl. Math. **34**, 1981, 29-51.

[3] F. John, *Nonlinear wave equations, Formation of singularities*, Pitcher Lectures in the Math. Sci. (Lehigh Univ.), Amer. Math. Soc., 1990.

[4] K. Kubota, K. Yokoyama, *Global existence of classical solutions to systems of nonlinear wave equations with different speeds of propagation*, to appear in Japan. J. Math.

[5] H. Lindblad, *On the lifespan of solutions of nonlinear wave equations with small initial data*, Comm. Pure Appl. Math. **43**, 1990, 445-472.

[6] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer.