Dependence and Unicity of Meromorphic Mappings

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Introduction.

Let $\pi : X \to \mathbb{C}^m$ be a finite analytic covering space and M a projective algebraic manifold. Let $f_1, \cdots, f_l$ be dominant meromorphic mappings from X into M. Suppose that they have the same inverse images of given divisors on M. In this talk, we give conditions under which $f_1, \cdots, f_l$ are algebraically related. Roughly our result says that if these mappings satisfy the same algebraic relation at all points of the set of the inverse images of divisors and if the given divisors are sufficiently ample, then they must satisfy this relationship identically. We first give criteria for algebraic dependence under a condition on the existence of meromorphic mappings separating the fibers of $\pi : X \to \mathbb{C}^m$. Next, we give applications of these criteria. In particular, we give some conditions under which two holomorphic mappings are related by endomorphism of elliptic curves.

§1 Criteria for propagation of algebraic dependence.

We first give a definition of algebraic dependence of meromorphic mappings. We set $M^l = M \times \cdots \times M$ (l-times). For meromorphic mappings $f_1, \cdots, f_l : X \to M$, we define a meromorphic mapping $f_1 \times \cdots \times f_l : X \to M^l$ by

$$(f_1 \times \cdots \times f_l)(z) = (f_1(z), \cdots, f_l(z)), \quad z \in X - (I(f_1) \cup \cdots \cup I(f_l)),$$

where $I(f_j)$ are the indeterminacy loci of f_j. A proper algebraic subset Σ of M^l is said to be *decomposable* if, for some positive integer s not grater than l, there exist positive integers $l_1, \cdots, l_s$ with $l = l_1 + \cdots + l_s$ and algebraic subsets $\Sigma_j \subseteq M^{l_j}$ such that $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$. We denote by B the ramofication divisor of $\pi : X \to \mathbb{C}$.

Definition 1. Let S be an analytic subset of X. Nonconstant meromorphic mappings $f_1, \cdots, f_l : X \to M$ are said to be *algebraically dependent on* S if there exists a proper algebraic subset Σ of M^l such that $(f_1 \times \cdots \times f_l)(S) \subseteq \Sigma$ and Σ is not decomposable. In this case, we also say that $f_1, \cdots, f_l$ *are* Σ*-related on* S.

Let $L \to M$ be an ample line bundle over M Let $D_1, \cdots, D_q$ be divisors in $|L|$ such that $D_1 + \cdots + D_q$ has only simple normal crossings, where $|L|$ is the complete linear system defined by L. Let $S_1, \cdots, S_q$ be hypersurfaces in X such that $\dim S_i \cap S_j \leq m-2$ for any $i \neq j$. We define a hypersurface S in X by $S = S_1 \cup \cdots \cup S_q$. Let E be an effective divisor on X, and let k be a positive integer. If $E = \sum_j \nu_j E'_j$ for distinct irreducible hypersurfaces E'_j in X and for nonnegative integers ν_j, then we define the support of E with order at most k by

$$\mathrm{Supp}_k\, E = \bigcup_{0<\nu_j\le k} E'_j.$$

Assume that $\mathrm{Supp}_{k_j} f_0^* D_j$ coincides with S_j for all j with $1 \le j \le q$, where k_j is a fixed positive integer. Let $\mathcal{F}$ be the set of all *dominant* meromorphic mappings $f : X \to M$ such that $\mathrm{Supp}_{k_j} f^* D_j$ is equal to S_j for each j with $1 \le j \le q$. Let $F_1, \cdots, F_l$ be big line bundles over M. We define a line bundles $\tilde{F}$ over M^l by

$$\tilde{F} = \pi_1^* F_1 \otimes \cdots \otimes \pi_l^* F_l,$$

where $\pi_j : M^l \to M$ are the natural projections on j-th factor. Let $\tilde{L}$ be a big line bundle over M^l. In the case of $\tilde{L} \ne \tilde{F}$, we assume that there exists a positive rational number $\tilde{\gamma}$ such that $\tilde{\gamma}\tilde{F} \otimes \tilde{L}^{-1}$ is big. If $\tilde{L} = \tilde{F}$, then we take $\tilde{\gamma} = 1$. Let $\mathcal{R}$ be the set of all hypersurfaces Σ in X such that $\Sigma = \mathrm{Supp}\, \tilde{D}$ for some $\tilde{D} \in |\tilde{L}|$ and Σ is not decomposable. We denote by B the ramification divisor of $\pi : X \to \mathbb{C}^m$.

Definition 2. Let Y be a compact complex manifold. We say that a meromorphic mapping $f : X \to Y$ *separates the fibers of* $\pi : X \to \mathbb{C}^m$, if there exists a point z in $\mathbb{C}^m - (\mathrm{Supp}\, \pi_* B \cup \pi(I(f)))$ such that $f(x) \ne f(y)$ for any distinct points $x,\ y \in \pi^{-1}(z)$.

Let s_0 be the sheet number of $\pi : X \to \mathbb{C}^m$. Assume that $f : X \to M$ separates the fibers of $\pi : X \to M$. Since L is ample, there exist a positive integer μ and a pair of sections $\sigma_0, \sigma_1 \in H^0(M,\ \mu L)$ such that a meromorphic function $f^*(\sigma_0/\sigma_1)$ separates the fibers of $\pi : X \to \mathbb{C}^m$ for all such mappings f. We denote by μ_0 the least positive integer among those $\mu' s$. We assume that there exists a line bundle, say F_0, in $\{F_1, \cdots, F_l\}$ such that $F_0 \otimes F_j^{-1}$ is either big or trivial for all j. Set $k_0 = \max_{1\le j\le q} k_j$. We define $L_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$L_0 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0+1} F_0 \right).$$

Then we have our basic result, from which we see that, if L_0 is sufficiently big, then the algebraic dependence on S propagates to the whole space X.

Lemma 1. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{F}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on S. If $L_0 \otimes K_M$ is big, then $f_1, \cdots, f_l$ are Σ-related on X.*

Now, let us consider a more general case. Let $L_1, \cdots, L_l$ be ample line bundles over M. Let $q_1, \cdots, q_l$ be positive integers and assume that $D_j = D_{j1} + \cdots + D_{jq_j} \in |q_j L|$ has only normal crossings, where $D_{jk} \in |L|$. Let Z be a hypersurface in X. Let $\mathcal{G}$ be a family of dominant meromorphic mappings $f: X \to M$ such that

$$\mathrm{Supp}_{k_j} f^* D_j = Z$$

for some $1 \le j \le l$. In the case where $L_j = L$ for all j, we define $G_0 \in \mathrm{Pic}(M) \otimes \mathbb{Q}$ by

$$G_0 = \left(\min_{1 \le j \le l} \left\{ \frac{q_j k_j}{k_j + 1} \right\} - 2\mu_0(s_0 - 1) \right) L \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0 + 1} F_0 \right).$$

Then we have the following:

Lemma 2. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on Z. If $G_0 \otimes K_M$ is big, then $f_1, \cdots, f_l$ are Σ-related on X.*

For $F \in \mathrm{Pic}(M) \otimes \mathbb{Q}$, we define $[F/L]$ by

$$[F/L] = \inf \left\{ \gamma \in \mathbb{Q};\ \gamma L \otimes F^{-1} \ \text{ is big} \right\}.$$

Set

$$n_1 = q_1 - [K_M^{-1}/L] - 2\mu_0(s_0 - 1) \qquad \text{and} \qquad n_j = q_j - [K_M^{-1}/L] \ \ (2 \le j \le l).$$

We also set

$$p_j = \frac{q_j k_j}{1 + k_j} - [K_M^{-1}/L] - 2\mu_0(s_0 - 1)$$

for all j with $1 \le j \le l$ and $e_0 = 2\mu_0(s_0 - 1) + 1$. Then we have the following criterion, which is a corollary of Lemma 2:

Corollary. *Let $f_1, \cdots, f_l$ be arbitrary mappings in $\mathcal{G}$ and $\Sigma \in \mathcal{R}$. Suppose that $f_1, \cdots, f_l$ are Σ-related on Z. If all $n_j > 0$ and if*

$$p_1 - \frac{\tilde{\gamma} l k_0}{k_0 + 1}[F_1/L] + \sum_{j=2}^{l} \left(n_1 p_j - \frac{\tilde{\gamma} l e_0 k_0}{n_j (k_0 + 1)} [F_j/L] \right) > 0,$$

then $f_1, \cdots, f_l$ are Σ-related on X.

Remark. We give a remark on the assumptions in the above lemmas. In Lemma 1, we assume that $D_1, \cdots, D_q$ are linearly equivalent. We consider the case where D_i and D_j are not linearly equivalent for some pair $(i,\ j)$ but all the Chern classes $c_1([D_j])$ are identical. In this case, the conclusion of the Lemma 1 remains valid provided that the line bundle

$$\left(\sum_{j=1}^{q} \frac{k_j}{k_j + 1} - 2\mu_0(s_0 - 1) \right) [D_1] \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0 + 1} F_0 \right)$$

is ample. We next consider Lemma 2. In the case where L_i and L_j are not same for some i and j but all the Chern classes $c_1(L_j)$ are identical, the conclusion of Lemma 2 is still valid if the line bundle

$$\left(\min_{1\le j\le l}\left\{\frac{q_j k_j}{k_j+1}\right\} - 2\mu_0(s_0-1)\right) L_1 \otimes \left(-\frac{\tilde{\gamma} l k_0}{k_0+1} F_0\right)$$

is ample.

§2. Unicity theorems for meromorphic mappings.

In this section we give some unicity theorems as an application of criteria for dependence by taking line bundles F_j of special type. Let $\Phi : M \to \mathbb{P}_n(\mathbb{C})$ be a meromorphic mapping with $\operatorname{rank}\Phi = \dim M$. We denote by H the hyperplane bundle over $\mathbb{P}_n(\mathbb{C})$. Now let $l = 2$ and take $F_1 = F_2 = \Phi^* H$. We also take $\tilde{L} = \tilde{F}$. Then we see

$$L_0 = \left(\sum_{j=1}^{q} \frac{k_j}{k_j+1} - 2\mu_0(s_0-1)\right) L \otimes \left(-\frac{2k_0}{k_0+1}\Phi^* H\right).$$

A set $\{D_j\}_{j=1}^{q}$ of divisors is said to be *generic with respect to* f_0 *and* Φ provided that

$$R_j := f_0(X - I(f_0)) \cap \operatorname{Supp} D_j \cap \{w \in M;\ \operatorname{rank} d\Phi(w) = \dim M\} \ne \emptyset$$

for at least one j with $1 \le j \le q$. We assume that $\{D_j\}_{j=1}^{q}$ is generic with respect to f_0 and Φ. Let $\mathcal{F}_1$ be the set of all mappings $f \in \mathcal{F}$ such that $f = f_0$ on S. Then we have the following unicity theorems by Lemma 1:

Theorem 1. *Suppose that* $L_0 \otimes K_M$ *is big. Then the family* $\mathcal{F}_1$ *contains just one mapping* f_0.

We next consider the case $\dim M = 1$. Assume that M is a compact Riemann surface with genus g_0. In the case $g_0 = 0$, we have the following unicity theorem for meromorphic functions on X by Theorem 1 as follows:

Theorem 2. *Let* $f_1,\ f_2 : X \to \mathbb{P}_1(\mathbb{C})$ *be nonconstant holomorphic mappings. Let* $a_1, \cdots, a_d$ *be distinct points in* $\mathbb{P}_1(\mathbb{C})$. *The following hold.*

(1) *Suppose that* $\operatorname{Supp} {f_1}^* a_j = \operatorname{Supp} {f_2}^* a_j$ *for all* j. *If* $d \ge 2s_0 + 3$, *then* f_1 *and* f_2 *are identical on* X.

(2) *Suppose that* $\operatorname{Supp}_1 {f_1}^* a_j = \operatorname{Supp}_1 {f_2}^* a_j$ *for all* j. *If* $d \ge 4s_0 + 3$, *then* f_1 *and* f_2 *are identical on* X.

§3. Holomorphic mappings into smooth elliptic curves.

Let E be a smooth elliptic curve and let $f_1,\ f_2 : X \to E$ be nonconstant holomorphic mappings. We consider the problem to determine the condition which yields

$f_2 = \varphi(f_1)$ for an endomorphism φ of the abelian group E. We first note the following fact: If $f : X \to E$ separates the fibers of $\pi : X \to \mathbb{C}^m$, then we can take $\mu_0 = 2$. We denote by $[p]$ the point bundle determined by $p \in E$. Let $F_1 = F_2 = [p]$. Let $\varphi \in \mathrm{End}(E)$ and consider a curve

$$\tilde{S} = \{(x,\ y) \in E \times E;\ y = \varphi(x)\}$$

in $E \times E$. Let $\tilde{L}$ be the line bundle $[\tilde{S}]$ determined by $\tilde{S}$. In this section, $\tilde{\gamma}$ denotes the infimum of rational numbers such that $\gamma\tilde{F} \otimes [\tilde{S}]^{-1}$ is ample. Then we have $\tilde{\gamma} = \deg\varphi + 1$. This result proved by T. Kastura. By making use of Theorem 1, we have the following theorem:

Theorem 3. *Let* $f_1,\ f_2$ *and* φ *be as above. Let* $D_1 = \{a_1, \cdots, a_d\}$ *be a set of* d *points and* φ *a endomorphism of* E. *Set* $D_2 = \varphi(D_1)$. *Assume that the number of points in* D_2 *is also d. Suppose that* $\mathrm{Supp}_k\ f_1^* D_1 = \mathrm{Supp}_k\ {f_2}^* D_2$ *for some* k. *If* $d > 2(\deg\varphi + 1) + 8(s_0 - 1)(1 + k^{-1})$, *then* $f_2 = \varphi(f_1)$.

In the case where $\sharp D_2 < d$, we have the following theorem by Corollary of Lemma 2:

Theorem 4. *Let* $f_1,\ f_2 : \mathbb{C}^m \to E$ *be as above. Let* $D_1 = \{a_1, \cdots, a_d\}$ *be a set of* d *points and* $\varphi \in \mathrm{End}(E)$. *Set* $D_2 = \varphi(D_1)$. *Assume that the number of points in* D_2 *is* d'. *Suppose that* $\mathrm{Supp}_1\ f_1^* D_1 = \mathrm{Supp}_1\ f_2^* D_2$. *If* $dd' > (d + d')(\deg\varphi + 1)$, *then* $f_2 = \varphi(f_1)$.

The following unicity theorem is a direct conclusion of Theorem 3:

Theorem 5. *Let* $a_1, \cdots, a_d$ *be distinct points in* E. *Let* $f_1,\ f_2 : X \to E$ *be nonconstant holomorphic mappings. Suppose that* $\mathrm{Supp}_1\ f_1^* a_j = \mathrm{Supp}_1\ f_2^* a_j$ *for all* j. *If* $d > 16 s_0 - 12$, *then* f_1 *and* f_2 *are identical on* X.

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