Analytic smoothing effects for a class of dispersive equations

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1 Introduction and main result

Let m be an integer greater than or equal to 2. Let $P(y, D_y)$ be an linear differential operator in $\mathbb{R}$,

$$P(y, D_y) = D_y^m + \sum_{0 \le l \le m-1} c_l(y) D_y^l, \tag{1}$$

where $D_y = \frac{1}{i}\frac{\partial}{\partial y}$ and $c_l(y)$ are analytic functions in $\mathbb{R}$.

We shall make the following assumption.

One can find positive constants C_0, R_0, K_0 and $\sigma_0 \in (0,1)$ such that for $y \in \mathbb{R}, |y| > R_0$, and $k \in \mathbb{N} \cup \{0\}$,

$$\sum_{0 \le l \le m-1} |D_y^k c_l(y)| \le C_0 \frac{K_0^k k!}{|y|^{1+\sigma_0+k}}. \tag{2}$$

In our case the principal symbol for $P(y, D_y)$ is $p(y,\eta) = p(\eta) = \eta^m$. Let $\rho = (y, \eta) \in T^*\mathbb{R}\backslash 0$, and let $(Y(s; y, \eta), \Theta(s; y, \eta))$ be the solution to the equation,

$$\begin{cases} \frac{d}{ds}Y(s) &= \frac{\partial p}{\partial \eta}(Y(s), \Theta(s)), \quad Y(0) = y, \\ \frac{d}{ds}\Theta(s) &= -\frac{\partial p}{\partial y}(Y(s), \Theta(s)), \quad \Theta(0) = \eta. \end{cases} \tag{3}$$

Therefore

$$\begin{cases} Y(s) &= y + ms\eta^{m-1}, \\ \Theta(s) &= \Theta(0) = \eta. \end{cases} \tag{4}$$

Let us introduce a space of the initial data,

$$\Gamma_{\rho_0} = \{Y(s; y_0, \eta_0) \in \mathbb{R}, s \ge 0\}. \tag{5}$$

$$X_{\rho_0} = \{v \in L^2(\mathbb{R}); \exists \delta_0 > 0, e^{\delta_0 |y|^{\frac{1}{m-1}}} v(y) \in L^2(\Gamma_{\rho_0})\}. \tag{6}$$

For $u_0 \in L^2(\mathbb{R})$ let $u(t, \cdot) \in C(\mathbb{R}; L^2(\mathbb{R}))$ be the solution of the initial value problem,

$$\begin{cases} D_t u + P(y, D_y)u = 0, \\ u|_{t=0} = u_0(y). \end{cases} \tag{7}$$

Theorem 1 *Let $P(y, D_y)$ be defined in* (1) *satisfying* (2) *and $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}\backslash 0$.*

Let $u_0 \in L^2(\mathbb{R})$ be in X_{ρ_0}. Then for all $t < 0$ ρ_0 does not belong to the analytic wave front set $WF_A[u(t, \cdot)]$ of the solution $u(t, \cdot)$ for (7).

In fact we can extend this result in more general cases. The speaker will show the extension of this result in the talk.

Our approach is based on FBI transform. The speaker will explain the way to apply the theory of FBI transform into the study of the smoothing effects.

2 Analytic wave front set and FBI transform

Let $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}\backslash 0$. Let $\varphi(x, y)$ be a holomorphic function in a neighborhood $U_0 \times V_{y_0}$ of $(0, y_0)$ in $\mathbb{C} \times \mathbb{C}$ which satisfies

$$\frac{\partial \varphi}{\partial y}(0, y) = -\eta_0, \tag{8}$$

$$\operatorname{Im} \frac{\partial^2 \varphi}{\partial y^2}(0, y) > 0, \tag{9}$$

$$\frac{\partial^2 \varphi}{\partial x \partial y}(0, y) \neq 0. \tag{10}$$

For above $\varphi(x, y)$ we can define,

$$\Phi(x) = \max_{y \in V_{y_0}}(-\operatorname{Im} \varphi(x, y)), \tag{11}$$

for $x \in U_0$.

Let $a(x, y, \lambda) = \sum_{k \geq 0} a_k(x, y)\lambda^{-k}$ be a analytic symbol of order zero, elliptic in a neighborhood of $(0, y_0)$. Let $\chi \in C_0^\infty$ be a cutoff function with support in a neighberhood of y_0, $0 \leq \chi \leq 1$, and $\chi \equiv 1$ near y_0.

The FBI transform of a distribution $u \in \mathcal{D}'(\mathbb{R})$ is defined by

$$Tu(x, \lambda) = \langle \chi(\cdot)u, e^{i\lambda\varphi(x,\cdot)}a(x, \cdot, \lambda)\rangle, \quad \lambda > 1. \tag{12}$$

Assume that $u(t, \cdot)$ is a element of a family of distribution on $\mathbb{R}$ depending of a real parameter t. Let $t_0 \in \mathbb{R}$. We shall say that a point $\rho_0 \in T^*\mathbb{R}\backslash 0$ does not belong to the locally uniform analytic wave front set $\widetilde{WF_A}[u(t_0, \cdot)]$ if there exist an FBI transform T, positive constants $C, \mu, \lambda_0, \varepsilon$, and a neighborhood U_0 of 0 such that

$$e^{-\lambda\Phi(x)}|Tu(t, x, \lambda)| \leq Ce^{-\mu\lambda}, \quad \text{for } \forall x \in U_0, \forall \lambda \geq \lambda_0, \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon). \tag{13}$$

References

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