Periodic solutions of the Korteweg-de vries equation driven by white noise

The stochastic Korteweg-de Vries equation arises when modelling the propagation of weakly nonlinear waves in a noisy plasma. It is also of interest in any circumstances where the Korteweg-de Vries equation is used, the stochastic forcing may represent terms that have been neglected in the derivation of this ideal model. Lastly, many works are devoted to the derivation of a forced Korteweg-de Vries equation, and in that case, it can sometimes be reasonable to assume that the forcing contains a stochastic part.

When written in a convenient set of coordinates, the stochastic Korteweg-de Vries equation has the form

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} = \Phi\frac{\partial^2 B}{\partial t \partial x}\ , \tag{1}$$

where $u(t,x)$ is a random process, Φ is a linear operator and B is a two parameter brownian motion, that is a zero mean gaussian process whose correlation function is given by

$$\mathbb{E}(B(t,x)B(s,y)) = (t \wedge s)(x \wedge y).$$

The talk will explain a result obtained in a joint work with A. Debussche and Y. Tsutsumi, concerning the existence and uniqueness of strong (pathwise) solutions for equation (1), in the case where $x \in \mathbb{T} = [0, 2\pi]$ with periodic boundary conditions.

If the covariance operator Φ is described by a kernel $\mathcal{K}(x,y)$, the correlation function of the noise is then given by

$$\mathbb{E}(\Phi\frac{\partial^2 B}{\partial t \partial x}(t,x)\Phi\frac{\partial^2 B}{\partial t \partial x}(s,y)) = c(x,y)\delta_{t-s}$$

for $t, s \geq 0$, $x, y \in \mathbb{T}$. Here δ is the Dirac function and

$$c(x,y) = \int_{\mathbb{T}} \mathcal{K}(x,z)\mathcal{K}(y,z)dz.$$

The case where Φ is the identity operator on $L^2(\mathbb{T})$ corresponds to the space time white noise : $c(x,y) = \delta_{x-y}$. We wish to consider a noise which is as close as possible to this case.

We may write equation (1) in Itô form

$$du + (\partial_x^3 u + u\partial_x u)dt = \Phi dW, \tag{2}$$

where $W(t) = \partial B/\partial x$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. Note that W may also be written as $W(t,x) = \sum_{k=0}^{\infty} \beta_k(t) e_k(x)$ where $(e_k)_{k\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\mathbb{T})$ and $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of mutually independant real brownian motions in a fixed probability space.

Equation (2) is supplemented with the initial condition

$$u(0,x) = u_0(x). \tag{3}$$

The equation may then be written by using the Duhamel formula

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\left(u\partial_x u\right)(s)ds + \int_0^t U(t-s)\Phi dW(s).$$

Here, $U(t) = e^{-t\partial_x^3}$ is the Airy group. Using the unitarity of $U(t)$ in any Sobolev space $H^s(\mathbb{T})$, it is not difficult to see that the Ito stochastic integral

$$w(t) = \int_0^t U(t-s)\Phi dW(s)$$

lies almost surely in $H^s(\mathbb{T})$ only if Φ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$. For $\Phi = Id$, this holds true if and only if $s < -1/2$. Hence we need to work with spatially less regular solutions as possible. For that purpose, we use Bourgain's method. The difficulty in applying the method lies in the lack of regularity in time of the brownian motion, which imposes the use of Besov's spaces.

We define

$$B^\sigma_{2,1}(\mathbb{T}) = \left\{f, \|f\|_{B^\sigma_{2,1}} < +\infty\right\}$$

with

$$\|f\|_{B^\sigma_{2,1}} = \sum_{n=1}^{+\infty} 2^{\sigma n}\left(\sum_{n'=2^{n-1}}^{2^{n+1}} |\hat{f}(n')|^2\right)^{1/2}.$$

The following theorem is obtained.

THEOREM. Assume that Φ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ into $H^s(\mathbb{T})$ for some $s > -1/2$, and assume that $u_0(\omega, x)$ is such that $u_0 \in B^\sigma_{2,1}(\mathbb{T})$ almost surely, for some σ with $-1/2 \leq \sigma < s$. Then for a.e. ω, there is a $T_\omega > 0$, and a unique solution $u(t)$ of the initial-boundary value problem (2)-(3) on $[0, T_\omega]$, which satisfies $u \in C([0, T_\omega]; B^\sigma_{2,1}(\mathbb{T}))$ (the uniqueness holds in some space $X \subset C([0, T_\omega]; B^\sigma_{2,1}(\mathbb{T}))$.