DECOMPOSITION OF MANIFOLDS AND A SPLITTING FORMULA FOR A SPECTRAL FLOW

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1. Introduction

It is well known that the non-trivial component $\widehat{\mathcal{F}}_*$ of the space $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_- \cup \widehat{\mathcal{F}}_* \cup \widehat{\mathcal{F}}_+$ of bounded selfadjoint Fredholm operators (on a complex or a real Hilbert space) is a classifying space for the K-group, K^{-1} in the complex case and that of KO^{-7} in the real case ([AS]). Also we know, by Bott periodicity theorem, that the fundamental group of these spaces is isomorphic to $\mathbb{Z}$. This isomorphism is given by the quantity "**Spectral flow**", $\mathbf{Sf} : \pi_1(\widehat{\mathcal{F}}_*) \to \mathbb{Z}$, although it is not stated explicitly in this paper [AS].

Since the *Spectral flow* was treated in the paper [APS], it appears in the various theory where it plays important roles, for example, in the study of spectral analysis of Dirac operators or in the theory of Floer homology for the study of low dimensional manifolds, and so on.

We have an *intuitive* understanding of the spectral flow by saying that it is the difference of the **net numbers** of the eigenvalues of the selfadjoint Fredholm operators in the family which change signs from minus to plus and from plus to minus, when the parameter of the family goes from 0 to 1. However in the paper [Ph] for the first time it was given a rigorous definition of the spectral flow for not only continuous loops in the space $\widehat{\mathcal{F}}_*$ but also for arbitrary continuous paths in $\widehat{\mathcal{F}}_*$ and was proved that the quantity is a homotopy invariant of continuous paths in the space $\widehat{\mathcal{F}}_*$ with the fixed end points and it satisfies the additivity under catenation of the continuous paths. Thus the spectral flow is not only a spectral invariant but also a homotopy invariant, in so far as we treat it in the framework of the space $\widehat{\mathcal{F}}_*$.

Nowadays there are many types of formulas including various *Spectral flows* corresponding to families of Fredholm operators which are mostly not bounded operators, because simply they are families of differential operators. However in some cases the **continuous** family of such unbounded Fredholm operators can be interpreted as a continuous path in the space $\widehat{\mathcal{F}}_*$ as treated in the papers [FO1], [BF1] and [CP]. Also the theorems in [Fl], [Yo] and [Ni] can be interpreted in the framework of the space $\widehat{\mathcal{F}}_*$. See also [Ta], [Ge], [CLM2], [OF] and [DK] and others. It is not clear for me whether any "*continuous*" families of Fredholm operators, especially a family of unbounded selfadjoint Fredholm operators with varying domains of definition, can be interpreted in the framework of the space $\widehat{\mathcal{F}}_*$.

The main purpose of my talk is to explain a **General Splitting Formula for a Spectral Flow** *of a family* $\{A + C_t\}_{t\in[0,1]}$ *(*C_t *is of zeroth order) of first order selfadjoint elliptic differential operators on a closed manifold in the framework of the space* $\widehat{\mathcal{F}}_*$.

When we decompose the closed manifold M into two parts $M_\pm$ by a hypersurface Σ, $M = M_- \cup M_+$, $M_- \cap M_+ = \partial M_\pm = \Sigma$, then we also have a family of symmetric elliptic operators on each part $M_\pm$ simply by the restriction, and at this level it is nothing more to have two families of symmetric operators defined on manifolds with boundary. This means there are no natural choices of boundary conditions among elliptic boundary conditions under which the family becomes a family of selfadjoint Fredholm operators. Under this circumstances it will not be apparent whether the spectral flow of the family $\{A + C_t\}$ on the whole manifold is expressed as a sum of two spectral flows obtained by even imposing a *suitable* selfadjoint elliptic boundary condition on each part and adding a correction term which solely depends on the boundary manifold Σ, since partly by the reason that we have no local representations of the spectral flow.

To formulate our splitting formula in this situation, we shall take a following way, that is, firstly we construct a two-parameter family $\{\mathcal{A}_{s,t}\}_{s,t\in[0,1]}$ of selfadjoint Fredholm operators on M with $\mathcal{A}_{t,t} = A + C_t$. Then by noting the homotopy invariance and the additivity under catenation of the spectral flow, the sum of two spectral flows of $\{\mathcal{A}_{s,0}\}_{s\in[0,1]}$ and $\{\mathcal{A}_{1,t}\}_{t\in[0,1]}$ coincides with the spectral flow of the original family. From the definition, it is not true, but it *looks like* that the families $\{\mathcal{A}_{s,0}\}_{s\in[0,1]}$ and $\{\mathcal{A}_{1,t}\}_{t\in[0,1]}$ are defined on the each component respectively. In general it is not clear that each of these two spectral flows coincides with a spectral flow of a family obtained by imposing a suitable selfadjoint elliptic boundary condition on the restriction of $\{A + C_t\}$ to each part. Then secondly we explain that if the operators in the family are a **product form** near the separating hypersurface, then there is a selfadjoint elliptic boundary condition on each part by which the restrictions of the operators to each part become a family of selfadjoint Fredholm operators and their spectral flows coincide with each of $\{\mathcal{A}_{s,0}\}_{s\in[0,1]}$ and $\{\mathcal{A}_{1,t}\}_{t\in[0,1]}$. Moreover it turned out that these boundary conditions reflect the influence from one side to other side of $M_\pm$ in a natural way. We verify these completely in the framework of the space $\widehat{\mathcal{F}}_*$. Our **General Spectral Flow Formula**([BF1]) and a **Reduction Theorem**([BFO]) of the Maslov index in the infinite dimension play a role as a bridge connecting these two spectral flows for the family of operators of a product form near the separating hypersurface Σ.

There are several similar formulas already treated in the papers [Ta], [DK], [CLM2] and others. Here I would like to emphasize that we can admit the non-invertible end points in the family $\{A + C_t\}$ because we base on the rigorous definition as a homotopy invariant of the spectral flow given in [Ph] together with that of the Malsov index which is valid without any assumptions at the end points and our method will explain in some extent what kinds of conditions to the operators in the family we need to prove such a splitting formula. Moreover, since for a family of the operators with product form structure near the separating hypersurface, the space of our boundary values $\boldsymbol{\beta} = D_{max}/D_{min}$ is identical for any length of the "neck", we expect this property also will give us a reduction of our splitting formula under taking *adiabatic limit* in within our framework.

Finally I remark that although there is a deep theory of pseudo-differential operators with transmition property including the theory of **Calderón projector** ([Ho2]), here I avoid the use of Calderón projector in the L_2-framework to treat with the **Cauchy data space**. Because it will make things *confusing* to use the Calderón projector in the L_2-framework from the beginning, and we miss the role of the product form assumption on the operators in our formula. However I must employ two facts without proofs from the pseudo-differential operator theory:

(a) the space of boundary values $\boldsymbol{\beta} = D_{max}/D_{min}$ of sections in the maximum domain in L_2-space of a first order selfadjoint elliptic differential operator on a manifold with boundary Σ is included in the Sobolev space of order $-\frac{1}{2}$ on the boundary manifold ([Ho1]).

(b) if a first order symmetric elliptic differential operator is a product form near the boundary manifold (= separating hypersurface in our setting), then the closed extension defined by the **Atiyah-Patodi-Singer boundary condition** is self-adjoint and satisfy **coercive estimate**, which we need to determine the space $\boldsymbol{\beta}$.

2. Notations and Main Theorem

2.1. **Fredholm-Lagrangian-Grassmannian.** Let $(H, \langle\cdot,\cdot\rangle, \omega)$ be a symplectic (real and separable) Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the symplectic form ω. In the theory below we do not replace the symplectic form ω after once it was introduced in the real Hilbert space H, but we may always assume that there exists a bounded operator $J: H \to H$ such that $\omega(x,y) = \langle Jx, y\rangle$ for any $x, y \in H$ and $J^2 = -\operatorname{Id}$ by replacing the inner product by another one which defines an equivalent norm on H. So we can assume from the beginning the following "compatible" conditions among ω, the inner product and the almost complex structure J:

$$\begin{aligned} \langle Jx, Jy\rangle &= \langle x, y\rangle \\ \omega(Jx, Jy) &= \omega(x,y) \quad \text{for all } x, y \in H. \end{aligned}$$

Let $\Lambda(H)$ denotes the set of all Lagrangian subspaces of H. Each Lagrangian subspace is closed, and the topology of $\Lambda(H)$ is given by embedding it into the space $\mathcal{B}(H)$ of bounded linear operators on H as the orthogonal projection operator onto the Lagrangian subspace.

Definition 2.1. Let $\lambda \in \Lambda(H)$, then the Fredholm-Lagrangian-Grassmannian of H with respect to λ is defined as

$$\mathcal{F}\Lambda_\lambda(H) := \{\mu \in \Lambda \mid (\mu, \lambda) \text{ is a Fredholm pair}\}.$$

The Maslov index in the infinite dimensional case, $\mathbf{Mas}(\{c(t)\}, \lambda) \in \mathbb{Z}$, for arbitrary continuous curves in $\mathcal{F}\Lambda_\lambda(H)$ with respect to the Maslov cycle $\mathfrak{M}_\lambda(H) = \{\mu \in \mathcal{F}\Lambda_\lambda(H) \mid \mu \cap \lambda \neq \{0\}\}$ is defined by the similar arguments in [Ph](see [BF1] and [FO2]).

2.2. **Symmetric Operators and Boundary Value Space.** Let H be a real separable Hilbert space and A a densely defined closed symmetric operator with the domain D_m. We denote the domain of the adjoint operator A^* by D_M.

Let $\boldsymbol{\beta}$ be the factor space of D_M by D_m, $\boldsymbol{\beta} = D_M/D_m$, and let $\gamma : D_M \to \boldsymbol{\beta}$ ($\gamma(x) = [x]$) be the projection map. The space $\boldsymbol{\beta}$ becomes a symplectic Hilbert space with the inner product induced by the graph norm

$$\langle x, y\rangle_G := \langle x, y\rangle + \langle A^*x, A^*y\rangle$$

on D_M and the symplectic form given by "*Green's form*"

$$\omega([x],[y]) := \langle A^*x, y\rangle - \langle x, A^*y\rangle \quad \text{for } x, y \in \boldsymbol{\beta}.$$

For $D_m \subset D \subset D_M$, we denote

$$A_D := A^*|_D,$$

then

$$\begin{aligned} A_D \text{ is closed symmetric} &\quad\Leftrightarrow\quad \gamma(D) \text{ is closed isotropic} \\ A_D \text{ is self-adjoint} &\quad\Leftrightarrow\quad \gamma(D) \text{ is Lagrangian} \end{aligned}$$

Definition 2.2. We call $\gamma(\operatorname{Ker} A^*)$ the **Cauchy data space**.

Now we make two Assumptions (A) and (B).

Assumption (A). We assume that A admits at least one self-adjoint Fredholm extension A_D with the domain of definition $D \subset D_M$ and A_D has the compact resolvent.

Then $\gamma(\operatorname{Ker}(A^*))$ is a Lagrangian subspace of $\boldsymbol{\beta}$ and $(\gamma(\operatorname{Ker} A^*), \gamma(D))$ is a Fredholm pair.

Assumption (B). We assume that there exists a continuous curve $\{C_t\}_{t\in[0,1]}$ in the space of bounded self-adjoint operators on H and that the operators $A^* + C_t - s$ for small s satisfy the condition that for any t

$$\operatorname{Ker}(A^* + C_t - s) \cap D_m = \{0\} \quad \text{for} |s| \ll 1.$$

Then the family of Lagrangian subspaces $\{\gamma(\operatorname{Ker}(A^* + C_t))\}$ is a continuous curve in $\mathcal{F}\Lambda_{\gamma(D)}(\boldsymbol{\beta})$.

Theorem 2.3 (General Spectral Flow Formula). *Let $\{A_D + C_t\}$ be a family of selfadjoint operators satisfying Assumption (A) and (B) above. Then we have*

$$\mathbf{Sf}(\{A_D + C_t\}) = \mathbf{Mas}(\{\gamma(S_t)\}, \gamma(D)).$$

2.3. **A Reduction Theorem for the Maslov Index.** Let $\boldsymbol{\beta}$ and L be two symplectic Hilbert spaces with the symplectic form $\omega_{\boldsymbol{\beta}}$ and ω_L. We assume that each space $\boldsymbol{\beta}$ and L is decomposed into a direct sum of two Lagrangian subspaces in the following way:

$$\boldsymbol{\beta} = \theta_- \oplus \theta_+$$

and

$$L = L_- \oplus L_+.$$

We also assume that there are injective maps with dense images

$$\mathbf{i}_- : \ \theta_- \to L_-$$

$$\mathbf{i}_+ : \ L_+ \to \theta_+$$

such that $\omega_L(\mathbf{i}_-(x), y) = \omega_{\boldsymbol{\beta}}(x, \mathbf{i}_+(y))$ for $x \in \theta_-$ and $y \in L_+$.

Then

Theorem 2.4 (Reduction Theorem). *There is a natural continuous map*

$$\ell : \ \mathcal{F}\Lambda_{\theta_-}(\boldsymbol{\beta}) \to \mathcal{F}\Lambda_{L_-}(H)$$

preserving the Maslov index:

$$\mathbf{Mas}(\{\mu_t\}, \theta_-) = \mathbf{Mas}(\{\ell(\mu_+)\}, L_-).$$

2.4. **Decomposition of manifolds and first order differential operators.** Let A be an elliptic and selfadjoint first order differential operator acting on a real vector bundle $\mathbb{E}$ on a closed manifold M. The selfadjointness of the operator A will be meant by fixing an L_2-inner product on the space of smooth sections $C^\infty(M,\mathbb{E})$. Let $\{C_t\}$ be a continuous family of selfadjoint bundle maps of $\mathbb{E}$, and we regard them with the same notation as a continuous family of zeroth order operators on $L_2(M,\mathbb{E})$. Let Σ be a hypersurface, which separates the manifold M into two parts $M_\pm$ with the commom boundary $\Sigma = \partial M_- = \partial M_+$. We assume that the operators $A + C_t + s$ ($|s| << 1$) satisfy Assumption (B) above and we assume that these operators $A + C_s$ are all of the "product form" near the hypersurface Σ, that is, the operator $A + C_s$ has the following form on a cylindrical neighborhood $U(\Sigma) \cong (-1,1)\times\Sigma$ of Σ:

$$A + C_s = \sigma\left(\frac{\partial}{\partial\tau} + B_s\right)$$

where $\tau \in (-1,1)$, σ is a bundle map of $\mathbb{E}|_\Sigma$, B_s is a selfadjoint elliptic operator on $\mathbb{E}|_\Sigma$, $\pi^*(\mathbb{E}|_\Sigma) \cong \mathbb{E}|_{U(\Sigma)}$, and $\pi : U(\Sigma) \to \Sigma$ is defined through a fixed identification $(-1,1)\times\Sigma \cong U(\Sigma)$. σ and B_s do not depend on the normal variable τ.

Let D_{min} be a subspace in $H^1(M,\mathbb{E})$ (the first order Sobolev space with values in $\mathbb{E}$) consisting of sections which vanish on Σ. Then the operator A defined on D_{min}, we denote it by $\mathcal{A}_0$, is a closed symmetric operator. We will denote the domain of the adjoint operator $(\mathcal{A}_0)^*$ by D_{max}. Here we are assuming that the hypersurface Σ **separates** the manifold M into two parts $M_\pm$ with common boundary $\Sigma = \partial M_\pm$, so D_{min} and D_{max} also decompose into two components $D^\pm_{min}$ and $D^\pm_{max}$ corresponding to submanifolds $M_\pm$:

$$\begin{aligned} D_m &= D_- \oplus D_+, \\ D_{max} &= D^-_{max} \oplus D^+_{max} \\ \mathcal{A}_0 &= \mathcal{A}_0^- \oplus \mathcal{A}_0^+, \end{aligned}$$

and $D^\pm_{max}$ is the domain of the each adjoint operator of $\mathcal{A}_0^\pm$.

Although graph inner products on D_{max} by operator $A + C_t$ is not identical, but the norms are all equivalent and the symplectic forms $\omega_\pm$ on $\boldsymbol{\beta}^\pm = D^\pm_{max}/D^\pm_{min}$ does not depend on the parameter.

Remark 2.5. If Σ is only orientable and does not separate M, that is, $M\backslash\Sigma$ is connected, then D_{min}(and also D_{max}) does not decompose in the above way, but $\boldsymbol{\beta} = D_{max}/D_{min}$ is a sum of two spaces $\boldsymbol{\beta}^\pm$ of boundary values taken from each side. However in this case the Cauchy data space $\gamma(\mathrm{Ker}(\mathcal{A}_0)^*)$ is not a sum of two Lagrangian subspaces in $\beta^\pm$.

Proposition 2.6 ([Ho1])**.** *$\boldsymbol{\beta}^\pm$ is a subspace in $H^{-1/2}(\Sigma,\mathbb{E}|_\Sigma)$, the Sobolev space on Σ of order $-1/2$. Also we have $\boldsymbol{\beta}^- \cap \boldsymbol{\beta}^+ = H^{1/2}(\Sigma,\mathbb{E}|_\Sigma)$.*

Let $\gamma_\pm$ be the map from $D^\pm_{max}$ to $\boldsymbol{\beta}^\pm$. The families $\{\Lambda^\pm_t\} = \{\gamma_\pm(\mathrm{Ker}((\mathcal{A}_0^\pm + C_t)^*))\}$ are continuous curves in $\Lambda(\boldsymbol{\beta}^\pm)$.

Theorem 2.7. *$\Lambda^-_t \oplus \Lambda^+_t \in \mathcal{F}\Lambda_{\boldsymbol{\delta}}(\boldsymbol{\beta})$, and $\mathbf{Sf}\{A + C_t\} = \mathbf{Mas}(\{\Lambda^-_t \oplus \Lambda^+_t\}, \boldsymbol{\delta})$ where*

$\boldsymbol{\delta} = \{(\varphi,\varphi) \in \boldsymbol{\beta}^-\oplus\boldsymbol{\beta}^+ \mid {}^\exists f \in H^1(M,\mathbb{E}), \gamma_\pm(f|_{M_\pm}) = \varphi\} \cong$ *diagonal of* $H^{1/2}(\Sigma)\oplus H^{1/2}(\Sigma)$.

Next let us consider the family of operators $\{T_s^-\}$ on M_- with the domain D_0,

$$D_0 = \{f \in H^1(M_-, \mathbb{E}|_{M_-}) \mid {}^\exists\varphi \in \Lambda_0^+, \gamma_-(f) = \varphi\} \tag{2.1}$$

and

$$\mathcal{T}_s^-(f) = (A + C_s)(f) \quad \text{on } M_-.$$

From the "*product form*" assumption and the Reduction Theorem 2.4, we know that $\gamma_-(D_0)$ is a Lagrangian subspace in $\boldsymbol{\beta}^-$. Hence we have

Proposition 2.8. *Each operator $\mathcal{T}_s^-$ on D_0 is a selfadjoint operator. So this implies that*

$$\|u\|_1 \leq c'(\|\mathcal{T}_s^- u\|_0 + \|u\|) \quad \textit{for } u \in D_0$$

with a uniform constant $c' > 0$.

Now we have the well-defined spectral flow for the continuous family $\{\mathcal{T}_s^- \circ \sqrt{1 + (\mathcal{T}_s^-)^2}^{-1}\}$:

Proposition 2.9.

$$\begin{aligned}\mathbf{Sf}(\{\mathcal{T}_s^-\}) \equiv \mathbf{Sf}(\{\mathcal{T}_s^- \circ \sqrt{1 + (\mathcal{T}_s^-)^2}^{-1}\}) &= \mathbf{Mas}(\{\Lambda_s^-\}, \gamma_-(D_0)) \\ &= \mathbf{Mas}(\{\Lambda_s^- \cap L_2(\Sigma)\}, \Lambda_0^+ \cap L_2(-\Sigma)).\end{aligned}$$

Similarly when we define the family of operators $\{\mathcal{T}_t^+\}$ on M_+ with the domain D_1 given by

$$D_1 = \{v \in H^1(M_+, \mathbb{E}|_{M_+}) \mid {}^\exists\varphi \in \Lambda_1^-, \gamma_+(v) = \varphi\}$$
$$\mathcal{T}_t^+(v) = (A + C_t)v \quad \text{on } M_+.$$

then

Proposition 2.10.

$$\begin{aligned}\mathbf{Sf}(\{\mathcal{T}_t^+\}) &= \mathbf{Mas}(\{\Lambda_t^+\}, \gamma_+(D_1)) \\ &= \mathbf{Mas}(\{\Lambda_t^+ \cap L_2(\Sigma)\}, \Lambda_1^- \cap L_2(-\Sigma)).\end{aligned}$$

Summing up these results we can now state our main Theorem.

Theorem 2.11 (Splitting Formula for a Spectral Flow).

$$\mathbf{Sf}\{A + C_t\} = \mathbf{Sf}\{\mathcal{T}_t^-\} + \mathbf{Sf}\{\mathcal{T}_t^+\}.$$

Of course this is not a unique way. For example, if we change the definition of the domains D_0 and D_1 by an obvious way, then we have a similar splitting formula.

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