STOCHASTIC ANALYSIS ON THE FIELD OF P-ADIC NUMBERS

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Abstract. After Evans suggested the significance of stochastic processes on the field of p-adics, Albeverio and Karwowski made a first major progress on p-adic stochastic analysis. They constructed stochastic proocesses by solving Kolmogorov equations. We will first construct a spatially inhomgeneous process by using Dirichlet space theory and give recurrence and transience criteria. These criteria will be akin to the standard comparison principle in the case that two comparable elliptic operators are given. Secondly, we will shed light on an advantage of stochastic integrals with respect to random walks in a restrictive class and introduce a framework of stochastic differential equation other than the one established by Kochubei. This study will be reminiscent of the existing stochastic analysis based on martingales.

§1. Introducntion.

In the theory elaborated by Albeverio and Karwowski (1994), an intuitively acceptable class of stochastic processes is introduced, the transition density function is derived and the corresponding Dirichlet space is described. In fact, by introducing any sequence $A = \{a(m)\}_{m=-\infty}^{\infty}$ satisfying

(i) $\quad a(m) \geq a(m+1)$,

they construct a symmetric Hunt process associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{Q}_p; \mu)$ determined by

$$\mathcal{E}(1_{B_1}, 1_{B_2}) = -2J(B_1, B_2) = -\frac{p^{K+L-m+1}}{p-1}(a(m-1) - a(m)),$$

where B_1 and B_2 stand for balls with radii p^K and p^L respectively, $\operatorname{dist}(B_1, B_2) = p^m$ and μ stands for the Haar measure on $\mathbb{Q}_p$.

Theorem (Yasuda, 1996). *The symmetric Markov process corresponding to $A = \{a(m)\}_{m=-\infty}^{\infty}$ is recurrent, if and only if* $\displaystyle\sum_{m=0}^{\infty} \frac{1}{p^m a(m)} = \infty$.

On the other hand, the α-stable process is characterized as the case that the Fourier transformation of the transition density p_t is given by $\exp(-t\|x\|_p^\alpha)$. Thanks to the additive characters χ_p, the γ-order derivative $D^\gamma u$ of locally constant function (for the definition, see e.g. Vladimirov, Volovich and Zelenov (1993)) u is describable by pseudo-differential operator and explicitly written as

$$D^\gamma u(x) = \int_{\mathbb{Q}_p} \|\xi\|_p^\gamma \hat{u}(\xi)\chi_p(-\xi x)\mu(d\xi) = \frac{p^\gamma - 1}{1 - p^{-\gamma-1}} \int_{\mathbb{Q}_p} \frac{u(x) - u(y)}{\|x-y\|_p^{\gamma+1}}\mu(dy),$$

where $\hat{u}(\xi)$ stands for the Fourier transformation $\displaystyle\int_{\mathbb{Q}_p} \chi_p(\xi x)u(x)\mu(dx)$ of the function u and $\gamma > 0$.

It is rather easy to see that if we choose $\displaystyle\int_{\mathbb{Q}_p} D^{\alpha/2}u(x)D^{\alpha/2}v(x)\mu(dx) \quad (\alpha > 0)$ as a Dirichlet form, then it generates Albeverio-Karwowski's symmetric Hunt process corresponding to the sequence $A = \{c(p,\alpha)p^{-\alpha m}\}_{m=-\infty}^{\infty}$ with a constant $c(p,\alpha)$ depending only on p and α, which is nothing but the α-stable process. However, we see that even though we simply choose a positive bounded function σ on $\mathbb{Q}_p$, the bilinear form $\displaystyle\int_{\mathbb{Q}_p} D^\alpha u(x)D^\alpha v(x)\sigma(x)\mu(dx)$ is no longer Markovian in general.

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In section 3, we will establish a method which provides us with a class of spatially inhomogeneous Hunt processes other than the one considered by Karwowski and Vilela-Mendes (1994), in the sense that the function σ controlling inhomogeneity is introduced in a different fashion. We propose a class of Dirchlet forms corresponding to spatially inhomogeneous processes which covers a family of Dirchlet forms described by modified derivatives similarly to the case of the α-stable process described by the $\alpha/2$-order derivative. For example, by introducing a modified γ-order derivative $D^{\gamma,\sigma}u(x) = \frac{p^{\gamma}-1}{1-p^{-\gamma-1}} \int_{\mathbb{Q}_p} \frac{u(x)-u(y)}{\|x-y\|_p^{\gamma+1}}\sigma(y)\mu(dy)$, we will see that $\mathcal{E}^{\gamma,\sigma}(u,v) = \int_{\mathbb{Q}_p} D^{\gamma,\sigma}u(x)D^{\gamma,\sigma}v(x)\sigma(x)\mu(dx)$ gives rise to a regular Dirichlet form which generates a spatially inhomogeneous process on $\mathbb{Q}_p$. However, it can not be obtained by the transformation by multiplicative functional presented by Fukushima, Oshima and Takeda (1994).

We will investigate recurrence and transience criteria for the Dirichlet form $\mathcal{E}^{(\alpha/2,\sigma)}(u,v)$ in comparison with the α-stable processes.

§2. Transience and recurrence of symmetric Hunt processes of jump type.

In this section, we start with a locally compact Hausdroff metric space X with a continuous exhaustion function ρ and consider a regular Dirichlet space $(\mathcal{E},\mathcal{F})$ on $L^2(X;\mu)$ which has a representation

$$\mathcal{E}(u,v) = \int_{X\times X} (u(x)-u(y))(v(x)-v(y))J(dx,dy)$$

with the symmetric non-negative Radon measure J on $X\times X$ satisfying $J(\{(x,y)|x=y\})=0$.

Assumption. *We assume that there exists a continuous exhaustion function ρ such that*

(A.1) $\inf\rho = 0$ *and* $\sup\rho > 1$,

(A.2) $1_{B(s)} \in \mathcal{F}$ *for* $\forall B(s) = \{z \mid \rho(z) < s\}$.

Then, we can set

$$j(r,s) = \mathcal{E}(1_{B(r)}, 1_{B(s)}) = 2J(B(r),B(s)^c) < \infty \qquad \text{for } s \geq r.$$

We will consider functions depending only on ρ. Let us introduce a notation Δ for a division $\Delta : 1 = \xi_0 < \xi_1 < \cdots < \xi_N = R$ of the interval $[1,R]$ and notation Σ for a positive sequence $\Sigma = \{\sigma_n\}_{n=0}^N$ so that the parameters of radial functions are in control. Let $\mathrm{Cap}(\overline{B(1)};B(R))$ denote the 0-order capacity between the compact set $\overline{B(1)}$ and the open set $B(R)$ (i.e., $\inf\{\mathcal{E}(u,u)|\ u\in\mathcal{F},\ u=1\ \mu$-a.e. on $\overline{B(1)}$ and $u = 0$ outside $B(R)\}$). For the definition of the recurrence for Dirichlet space, see e.g. Fukushima (1994/95).

Theorem 1. *If for any division Δ of the interval $[1,R]$ there exist sequences $\{f_\Delta(n)\}_{n=0}^N$ and $\{g_\Delta(n)\}_{n=0}^N$ such that $j(\xi_n,\xi_m) \leq f_\Delta(n)g_\Delta(m)$ $(n,m=0,\cdots N)$ and if*

$$\sup_R \sup_\Delta \sum_{k=0}^{N-1} \frac{1}{g_\Delta(k)}\left(\frac{1}{f_\Delta(k)} - \frac{1}{f_\Delta(k+1)}\right) = \infty,$$

then $(\mathcal{E},\mathcal{F})$ is recurrent.

Theorem 2. *Suppose that there exists a decreasing function f_R taking 1 on $\overline{B(1)}$ such that*

$$\mathrm{Cap}(\overline{B(1)}; B(R)) = \mathcal{E}(f_R(\rho), f_R(\rho))$$

for any $R > 1$. If for any division Δ of the interval $[1, R]$ there exist sequences $\{f_\Delta(n)\}_{n=0}^N$ and $\{g_\Delta(n)\}_{n=0}^N$ such that $j(\xi_n, \xi_m) \geq f_\Delta(n) g_\Delta(m)$ $(n, m = 0, \cdots N)$ and

$$\sup_R \sup_\Delta \sum_{n=0}^N \frac{1}{g_\Delta(n) f_\Delta(n)} < \infty,$$

then $(\mathcal{E}, \mathcal{F})$ is transient.

§3. A class of Dirichlet spaces on p-adic number field.

Theorem 3. *The pair $(\mathcal{E}^{(\gamma,\sigma)}, \mathcal{F}^{(\gamma,\sigma)})$ of the bilinear form $\mathcal{E}^{(\gamma,\sigma)}$ and its domain $\mathcal{F}^{(\gamma,\sigma)}$ is a regular Dirichlet form on $L^2(\mathbb{Q}_p; \sigma\mu)$.*

Theorem 4.

(i) *$\alpha \geq 1$ and σ is bounded $\Rightarrow$ $(\mathcal{E}^{(\alpha/2,\sigma)}, \mathcal{F}^{(\alpha/2,\sigma)})$ is recurrent,*

(ii) *$\alpha < 1$ and there exists a positive constant δ such that $\sigma \geq \delta$ on $\mathbb{Q}_p$ $\Rightarrow$ $(\mathcal{E}^{(\alpha/2,\sigma)}, \mathcal{F}^{(\alpha/2,\sigma)})$ is transient.*

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