LOCAL ENERGY DECAY FOR THE NONLINEAR DISSIPATIVE WAVE EQUATION IN AN EXTERIOR DOMAIN

TOKIO MATSUYAMA

1. Introduction

Let $\Omega = \mathbb{R}^N$ or $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be an unbounded domain lying outside of its compact and smooth boundary $\partial\Omega$ contained in the ball $B_{\rho_0} = \{x \in \mathbb{R}^N; |x| \leq \rho_0\}$ (if $N = 1$, we assume $\Omega = (-\infty, \infty)$ or $\Omega = (\rho_0, \infty)$). In $(0, \infty) \times \Omega$ we consider the initial-boundary value problem

$$\text{(P)} \qquad \begin{cases} u_{tt} - \Delta u + a(t,x)|u_t|^\beta u_t = 0, & (t,x) \in (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega, \end{cases}$$

where $a(t,x)$ is a nonnegative function in $[0,\infty) \times \overline{\Omega}$ and $\beta > 0$. When Ω is the whole space $\mathbb{R}^N$, we can drop the boundary condition.

We make the following hypotheses on $a(t,x)$ as follows.

Hyp.A. *$a(t,x)$ is a nonnegative function in $[0,\infty) \times \overline{\Omega}$ and belongs to $C^1([0,\infty) \times \overline{\Omega})$ such that*

$$|a_t(t,x)| + |\nabla a(t,x)| \leq Ca(t,x)$$

for some $C > 0$, where $\nabla f = (\partial_1 f, \partial_2 f, \cdots, \partial_N f)$, $\partial_j = \partial/\partial x_j$.

Now, assume that

$$\{u_0, u_1\} \in [H^2(\Omega) \cap H_0^1(\Omega)] \times [H_0^1(\Omega) \cap L^{2(\beta+1)}(\Omega)].$$

Then it is well known (cf. Lions and Strauss [5]) that if we assume Hyp.A, then the problem (P) has a unique global solution satisfying the following properties:

(i) $u \in C([0,\infty); H_0^1(\Omega))$ and u satisfies the energy identity

$$E(u(t)) + \int_s^t \int_\Omega a|u_t|^{\beta+2}\, dx d\tau = E(u(s)) \tag{1.1}$$

for $0 \leq s < t$.

(ii) $u \in C([0,T]; L^2(\Omega))$ for any $T > 0$.

(iii) u_{tt}, ∇u_t, Δu, $a|u_t|^\beta u_t \in L^\infty(0,T; L^2(\Omega))$ for any $T > 0$.

We see from the energy identity (1.1) that the energy decreases in $t > 0$. Thus a question naturally arises whether it decays or not. When Ω is the whole space $\mathbb{R}^N$, Mochizuki and Motai has proved in [10] that if $a(t,x) \geq a_0(1 + t + |x|)^{-\delta}$ $(0 \leq \delta < 1)$ and $0 < \beta \leq 2(1-\delta)/N$, then the energy decays like $\{\log(e+t)\}^{-\mu}$ $(0 < \mu < 2/\beta)$, and if $0 \leq a(t,x) \leq a_1(1 + |x|)^{-\delta}$ $(0 \leq \delta \leq 1)$ and $\beta > 2(1-\delta)/(N-1)$ $(N \geq 2)$, then the energy does not in general decay (see also Mochizuki [8] which the proof is not given in detail). The decay estimate in [10] can be immediately shown in any exterior domain with a compact boundary. On the other hand, it seems difficult to discuss the energy nondecay problem in exterior domains without any assumption on the shape and the more restrictive condition on $a(t,x)$. For the linear dissipative case in an exterior domain with the star-shaped boundary we should refer to the work of Mochizuki and Nakazawa [11] (see also the work of the author [6] in an exterior domain where the dissipation is effective around the boundary).

The object in this paper is to discuss the energy nondecay of the solution to the problem (P) and construct the global classical solutions, say $H^4 \times H^3$ class, with the typical case $\beta = 2m$ and $N = 3$. Contrary to the result of [10], if the boundary is star-shaped and $a(t,x) = O(|x|^{-\delta})$ for some $\delta > 1$ as $|x| \to \infty$, we can prove the energy nondecay for $N \neq 2$ and $\beta > 0$ (Theorem 2.1). In the category of classical solutions we can also prove that if Ω is the whole space $\mathbb{R}^3$ and $a(t,x)$ satisfies $a(t,x) = O(|x|^{-\delta} b(t))$ for some $\delta > 2$ and $b(t) = o(1)$ $(t + |x| \to \infty)$, then the local energy decays at a certain rate as $t \to \infty$ (Theorem 2.3). In the case where Ω is an exterior star-shaped domain, if $a(t,x)$ has a compact support, then we shall also show in Theorem 2.4 that the local energy decays exponentially to 0 as $t \to \infty$. Here, we should remark the result in [8] of the local energy decay, which showed that if $\delta + (N-1)\beta/2 > 1$, then the local energy decays to 0, but its rate was not determined. Now the existence of classical solutions was treated by many authors (see Nakao [13] and Sather [15] in a bounded domain, Shatah [16] in the whole space, and Hayashi [2] in the exterior domain outside the ball, Shibata and Tsutsumi [17] in exterior nontrapping domains etc). In any way it was required for the existence of globally in time small amplitude solutions that the initial data belongs to the higher order Sobolev spaces. Our result concerning the existence of the global C^2 solution in Theorem 2.2 is even new in the sense that the small amplitude solutions belong to the weaker class $H^4 \times H^3$ than the previous ones. Since the energy does not in general decay, we cannot expect to obtain the decay estimate and as a result, it seems difficult to prove the global existence of classical solutions by the usual energy method.

In the case when Ω is the whole space $\mathbb{R}^3$ or an exterior domain outside the star-shaped obstacle in $\mathbb{R}^3$, if $a(t,x) = O(|x|^{-\delta})$ for some $\delta > 1$ as $|x| \to \infty$, we shall use the weighted energy method to obtain the space-time integrability of $a|D_t^j u|^2$ $(j = 1,2,3,4)$ in Proposition 5.5, and as a result, we can deduce that the wave operator $\partial_t^2 - \Delta$ has a dissipative effect like $a_1(1+|x|)^{-\delta} u_t$ for some $a_1 > 0$ and $\delta > 1$. From this observation it follows that the nonlinear term like au_t^3 can be absorbed into the space-time estimates of $a|D_t^j u|^2$, which enables us to assure the existence of global classical solutions in Theorem 2.2. Roughly speaking, we can interpret the above mentioned phenomenon as follows; since the term $a|u_t|^\beta u_t$ has the nature of both dissipation and perturbation, the solutions to the problem (P) behave like the solutions to the free wave problem.

Notations

$$D_t = \frac{\partial}{\partial t}, \quad D_t^j = \frac{\partial^j}{\partial t^j}, \quad D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\left(\frac{\partial}{\partial x_3}\right)^{\alpha_3} \quad (|\alpha| = \alpha_1 + \alpha_2 + \alpha_3),$$
$$E(u(t)) = \frac{1}{2}\{\|u_t(t)\|^2 + \|\nabla u(t)\|^2\},$$
$$I_j^2 = E(D_t^j u(0)) = \frac{1}{2}\{\|D_t^{j+1}u(0)\|^2 + \|\nabla D_t^j u(0)\|^2\} \quad (j = 0, 1, 2, 3).$$

Here by $\|\cdot\|$ we denote the L^2 norm over Ω. By $H_\nabla(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ in the Dirichlet norm $\|\nabla\cdot\|$. We set

$$H_\nabla^2(\Omega) = \{u \in H_\nabla(\Omega)\,;\, \Delta u \in L^2(\Omega)\}.$$

For an interval I in $\mathbb{R}$, we denote by $C_B^k(I; X)$ the space of all X-valued bounded k-times continuously differentiable functions over I.

2. Statement of results

First, we shall state the energy nondecay of solutions to the problem (P). For this, let us assume

Hyp.B. *$a(t,x)$ satisfies*

$$0 \le a(t,x) \le a_1(1+|x|)^{-1-\delta} \quad \text{in } [0,\infty)\times\overline{\Omega}$$

for some $a_1 > 0$ and $\delta > 0$.

Hyp.C. *$\partial\Omega$ is star-shaped with respect to the origin.*

We now consider the initial-boundary value problem for the free wave equation with the same initial data as the problem (P):

$$(\mathrm{P})_0 \quad \begin{cases} w_{tt} - \Delta w = 0, & (t,x) \in (0,\infty)\times\Omega, \\ w(0,x) = u_0(x), \quad w_t(0,x) = u_1(x), & x \in \Omega, \\ w(t,x) = 0, & (t,x) \in (0,\infty)\times\partial\Omega. \end{cases}$$

It is well known (cf. Ikawa [3]) that if $\{u_0, u_1\}$ further belongs to $H^{k+1}(\Omega)\times H^k(\Omega)$, k being an integer with $k > [N/2]$, and satisfies the compatibility condition of order k, that is, $w_j \in H_0^1(\Omega)$ $(j = 0, 1, \ldots, k; w_0 = u_0,\ w_1 = u_1)$ and $w_{k+1} \in L^2(\Omega)$, where $\{w_j\}$ are defined inductively by

$$w_j = \Delta w_{j-2} \ (j = 2, 3, \cdots, k+1), \tag{2.1}$$

then the finite energy solution $w(t,x)$ belongs, in fact, to

$$X^k(0,\infty) \equiv \bigcap_{j=0}^{k} C^j([0,\infty); H^{k+1-j}(\Omega) \textstyle\bigcap H_0^1(\Omega)) \textstyle\bigcap C^{k+1}([0,\infty); L^2(\Omega))$$

and satisfies

$$\sup_{t\ge 0}\|w_t(t)\|_{L^\infty} \le C_k \equiv C(\|u_0\|_{H^{k+1}}, \|u_1\|_{H^k}). \tag{2.2}$$

Of course, when $\Omega = \mathbb{R}^N$, we need not take account of Hyp.C and the compatibility condition.

Our first result reads as follows.

Theorem 2.1. *Let $N \neq 2$. Suppose Hyp.A, B, C and $\{u_0, u_1\} \neq \{0, 0\}$. Assume further that $\{u_0, u_1\}$ belongs to $H^{k+1}(\Omega) \times H^k(\Omega)$, k being a positive integer with $k > [N/2]$, and satisfies the compatibility condition of order k in the sense of (2.1). If we take $\sigma \equiv \sigma(u_0, u_1) > 0$ so that*

$$\int_\sigma^\infty \int_\Omega a|w_t(t)|^{\beta+2}\, dx dt < 2^{\beta+2} E(u(0)),$$

then the energy $E(u^{(\sigma)}(t))$ of the solution $u^{(\sigma)}(t)$ to the problem (P) *with the initial data $\{u_0, u_1\}$ replaced by $\{w(\sigma), w_t(\sigma)\}$ never decays to 0 as $t \to \infty$.*

Remark. If we impose the more restrictive condition on $a(t, x)$ like

$$0 \leq a(t,x) \leq a_1(1 + t + |x|)^{-\delta} \quad \text{in } [0, \infty) \times \overline{\Omega} \tag{2.3}$$

for some $a_1 > 0$ and $\delta > 1$, we can easily see that the same assertion in Theorem 2.1 is also valid for $N \geq 1$ without any geometrical condition like Hyp.C.

Next we want to treat the existence of globally in time small amplitude solutions to the problem (P). For this, we shall pay attention to the typical power nonlinearity such as au_t^{2m+1} ($\beta = 2m$).

Let Ω be the whole space $\mathbb{R}^3$ or the exterior domain in $\mathbb{R}^3$ with a compact boundary. Assume $m = 1$, 2 or $m \geq 3$, which assures that the nonlinear dissipative term is C^3 class.

Then we consider the initial-boundary value problem

$$(\mathrm{P})_m \quad \begin{cases} u_{tt} - \Delta u + a(t,x) u_t^{2m+1} = 0, & (t,x) \in (0,\infty) \times \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega. \end{cases}$$

We assume that $\{u_0, u_1\} \in H^4(\Omega) \times H^3(\Omega)$ satisfies the compatibility condition of order 3;

$$D_t^j u(0, \cdot) \in H_0^1(\Omega) \quad (j = 0, 1, 2, 3) \quad \text{and} \quad D_t^4 u(0, \cdot) \in L^2(\Omega). \tag{2.4}$$

Furthermore, we impose the other assumptions as

Hyp.B′ *In addition to Hyp.*B*, we assume that $a(t, x)$ belongs to $C^3([0, \infty) \times \overline{\Omega})$ and satisfies*

$$\sum_{j+|\alpha| \leq 3} |D_t^j D_x^\alpha a(t,x)| \leq C a(t,x) \quad \text{in } [0,\infty) \times \overline{\Omega}$$

for some $C > 0$.

Then our second result reads as follows.

Theorem 2.2. *Let $N = 3$. Assume Hyp.*B, B′ *and* C. *Then, there exist positive constants ε_0 and C_0 having the following properties; if the initial data $\{u_0, u_1\} \in H^4(\Omega) \times H^3(\Omega)$ satisfies*

$$\|\nabla u_0\|_{H^3} + \sum_{j=0}^{3} \|D_t^{j+1} u(0)\|_{H^{3-j}} < C_0 \varepsilon_0,$$

$$\|u_t(0)\|_{L^\infty} + \|u_{tt}(0)\|_{L^\infty} + \|\nabla u_t(0)\|_{L^\infty} < \varepsilon_0,$$

and the compatibility condition of order 3 *in the sense of* (2.4)*, then the problem* $(\mathrm{P})_m$ *admits a unique solution* $u \in C^2([0,\infty)\times\overline{\Omega})$ *so that*

$$\sup_{t\ge 0}\left\{\|\nabla u(t)\|_{H^3}+\sum_{j=0}^{3}\|D_t^{j+1}u(t)\|_{H^{3-j}}\right\}\le C_0C_1\varepsilon_0,$$

$$\sup_{t\ge 0}\left\{\|u_t(t)\|_{L^\infty}+\|u_{tt}(t)\|_{L^\infty}+\|\nabla u_t(t)\|_{L^\infty}\right\}\le \varepsilon_0$$

for some positive constant C_1.

Remark. If $N=1$, the existence and uniqueness of the global classical solution can de discussed under the assumptions in Theorem 2.2 replaced by $\{u_0,u_1\}\in H^3(\Omega)\times H^2(\Omega)$. If $N=2$ or 3 and $a(t,x)$ satisfies the decay condition (2.3), then the same assertion holds without any geometric condition like Hyp.C.

Finally, let us state the results of the local energy decay. Setting

$$E_{loc,R}(t)=\frac{1}{2}\int_{\Omega(R)}\{|u_t(t)|^2+|\nabla u(t)|^2\}\,dx\quad(\Omega(R)\equiv\Omega\cap B_R),$$

we have the following decay estimates of $E_{loc,R}(t)$. Of course, if $\Omega=\mathbb{R}^3$, then $\Omega(R)$ should be replaced by B_R.

Theorem 2.3. *Let* Ω *be the whole space* $\mathbb{R}^3$ *and* $u(t,x)$ *the solution in the sense of Theorem* 2.2 *with* $\operatorname{supp}u_0\cup\operatorname{supp}u_1\subset B_R$ *for any fixed number* $R>0$. *Assume that* $b(t)$ *is a positive and decreasing function on* $[0,\infty)$ *belonging to* $C_B^3([0,\infty))$ *such that*

$$\sum_{j=0}^{3}|D_t^j b(t)|\le Cb(t)\ \text{on}\ [0,\infty)\qquad\text{and}\qquad b(\cdot)\notin L^1((0,\infty))\ \text{with}\ b(t)=o(1)\ (t\to\infty)$$

for some $C>0$. *If* $a(t,x)$ *satisfies*

$$(2.5)\qquad a(t,x)\not\equiv 0\quad\text{and}\quad 0\le a(t,x)\le b(t)(1+|x|)^{-1-\delta}\quad\text{in}\quad[0,\infty)\times\mathbb{R}^3$$

for some $\delta>1$, *then we have*

$$E_{loc,R}(t)\le C(R,\varepsilon_0)\max\{(1+t)^{-2(\delta-1)},b(t)^2\}$$

for all $t\ge 0$.

Theorem 2.4. *Assume that* Ω *is the whole space* $\mathbb{R}^3$ *or an exterior domain in* $\mathbb{R}^3$ *satisfying Hyp.C. Let* $u(t,x)$ *be the solution in the sense of Theorem* 2.2 *with* $\operatorname{supp}u_0\cup\operatorname{supp}u_1\subset\Omega(R)$ *for any fixed number* R *with* $R>\rho_0$. *Then there exists a number* $\varepsilon_1\equiv\varepsilon_1(R)>0$ *with* $\varepsilon_1\le\varepsilon_0$ *such that if, in particular,* $a(t,x)$ *has a compact support in* $\Omega(R)$ *and* $\sup_{t\ge0}\|u_t(t)\|_{L^\infty}\le\varepsilon_1$, *then we have*

$$E_{loc,R}(t)\le C(R,\varepsilon_0)E(u(0))e^{-\lambda_0(R)t}$$

for all $t\ge 0$, *where we set*

$$\lambda_0(R)=2(\lambda-C(R)a_1\varepsilon_1^{2m})>0.$$

for some $\lambda>0$ *independent of* R.

Remark. In the case where $a(t,x)$ does not have the compact support, if $a(t,x) = O(|x|^{-1-\delta}e^{-\mu t})$ $(t+|x| \to \infty)$ for some $\delta > 1/2$ and $\mu > 2\lambda$, then the conclusion in Theorem 2.4 can be replaced by

$$E_{loc,R}(t) \leq C(R,\varepsilon_0)E(u(0))e^{-2\lambda t}.$$

References

[1] D. Gilberg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd ed., Springer, 1983.

[2] N. Hayashi, *Global existence of small solutions to quadratic nonlinear wave equations in an exterior domain*, J. Func. Anal. **131** (1995), 302–344.

[3] M. Ikawa, Hyperbolic partial differential equations and wave phenomena, Transl. Math. Monogr., Vol. 189, Amer. Math. Soc., 2000.

[4] O. A. Ladyzhenskaya, The mathematical theory of viscous incompressible flow, Revised 2nd ed., New York: Gordon and Breach, 1969.

[5] J. L. Lions and W. A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math. France **93** (1965), 43–96.

[6] T. Matsuyama, *Asymptotic behaviour of solutions to the initial-boundary value problem with an effective dissipation around the boundary*, preprint (2000).

[7] S. Mizohata, The theory of partial differential equations, Cambridge Univ. Press, 1973.

[8] K. Mochizuki, *Decay and asymptotics for wave equations with dissipative term*, Lecture Notes in Phys., **39** 1975, Springer-Verlag, pp. 486–490.

[9] K. Mochizuki, Scattering theory for wave equations (in Japanese), Kinokuniya, 1984.

[10] K. Mochizuki and T. Motai, *On energy decay-nondecay problems for the wave equations with nonlinear dissipative term in* $\mathbb{R}^N$, J. Math. Soc. Japan **47** (1995), 405–421.

[11] K. Mochizuki and H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. RIMS, Kyoto Univ. **32** (1996), 401–414.

[12] C. Morawetz, *Exponential decay of solutions of the wave equations*, Comm. Pure Appl. Math. **19** (1966), 439–444.

[13] M. Nakao, *Existence of global classical solutions of the initial-boundary value problem for some nonlinear wave equations*, J. Math. Anal. Appl. **146** (1990), 217–240.

[14] M. Nakao, *Stabilization of local energy in an exterior domain for the wave equation with a localized dissipation*, J. Diff. Eq. **148** (1998), 388–406.

[15] J. Sather, *The existence of a global classical solution of the initial-boundary value problem for* $\square u + u^3 = f$, Arch. Rational Mech. Anal. **22** (1966), 129–135.

[16] J. Shatah, *Global existence of small solutions to nonlinear evolution equations*, J. Diff. Eq. **46** (1982), 409–425.

[17] Y. Shibata and Y. Tsutsumi, *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Z. **191** (1986), 165–199.

Department of Mathematics, Tokai University, Hiratsuka, Kanagawa 259-1292, Japan

E-mail address: `matsu@@sm.u-tokai.ac.jp`