Supersingular perturbations of self–adjoint Operators

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1. Introduction

In this talk, mainly, I am going to show how to define arbitrary rank one H_{-3}-perturbations of self–adjoint operators restricting our consideration to the case of semibounded (positive) operators [**7**]. Then I am going to make some comments for [**6, 8, 9**].

Suppose that A is a certain self–adjoint operator acting in the Hilbert space H. Then a finite rank perturbation of it is formally defined by the formula

$$A_V = A + V, \tag{1.1}$$

where V is a finite dimensional operator. The domains of the perturbed and unperturbed operators coincide if the perturbation is a bounded operator in the Hilbert space $V \in \mathcal{B}(H)$. But it is possible to consider more general perturbations determined by operators V acting in the scale of Hilbert spaces associated with the original operator A

$$V : H_2(A) \to H_{-2}(A). \tag{1.2}$$

In the latter case the perturbed operator can be defined using the form perturbation technique and the extension theory for symmetric operators. Really the operators A and A_V restricted to the domain

$$\mathrm{Dom}(A^0) = \{\psi \in \mathrm{Dom}(A) : \psi \in \mathrm{Ker}(V)\}$$

coincide. The perturbed operator cannot be determined uniquely in this case only a finite parameter family corresponding to the formal expression (1.1) can be established. We shall consider that V is of H_{-3}-perturbation, i.e.,

$$V : H_3(A) \to H_{-3}(A). \tag{1.3}$$

Really consider the one dimensional perturbation formally determined by

$$A_\alpha = A + \alpha\langle\varphi, \cdot\rangle\varphi, \tag{1.4}$$

where $\varphi \in H_{-3}(A) \setminus H_{-2}(A)$. The restriction of the operator A to the domain $\mathrm{Dom}(A^0) = \{\psi \in H_3(A) : \langle\varphi, \psi\rangle = 0\}$ is essentially self–adjoint. Every self–adjoint extension of this symmetric operator coincides with the original operator A. Therefore the operator corresponding to (1.4) cannot be defined in the original

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Hilbert space using standard methods of extension theory. The aim of the current paper is to define the operator A_α as a self–adjoint operator in a certain extended Hilbert space.

All self–adjoint extensions of this symmetric operator are rotationally invariant. Therefore no non-spherically symmetric point interaction for the Laplace operator in $\mathbf{R}^3$ can be constructed. Different modifications of this approach have been suggested. The first mathematically rigorous approach uses perturbations in Pontryagin spaces, where the perturbed operator has been defined in a certain extension of the original Hilbert space with non positive definite scalar product [**5, 11, 12, 13**]. Recently I. Andronov suggested using extended Hilbert spaces to model non-spherically symmetric scatterers in $\mathbf{R}^2$ [**3**]. This model is very similar to the generalized point interactions introduced by B. Pavlov [**2, 10**]. We are going to use some of the ideas suggested by I. Andronov to construct rank one H_{-3} perturbations.

2. The Hilbert Space

Our aim is to define the self–adjoint operator corresponding to (1.4), where A is a certain positive self–adjoint operator acting in the Hilbert space H and φ is an element from the space H_{-3} from the scale of Hilbert spaces associated with the operator A. In what follows we are going to consider the case where

$$\varphi \in H_{-3} \setminus H_{-2}. \tag{2.1}$$

It is natural to determine the operator A_α using the restriction-extension method. The operator A_α coincides with one of the extensions of the operator A^0, which is the restriction of the operator A to the set of functions u from the domain $\mathrm{Dom}(A)$ of the operator A satisfying the condition

$$\langle \varphi, u \rangle = 0. \tag{2.2}$$

Suppose that the operator A is considered as an operator in the original Hilbert space H. Then the domain of the operator coincides with the space H_2, $\mathrm{Dom}(A) = H_2$ and the operator A^0 is essentially self–adjoint if φ satisfies (2.1). The operator A^0 is not essentially self–adjoint for such φ only if the domain of the unperturbed operator A is a subset of H_3. Therefore let us consider the operator A as a self–adjoint operator acting in the Hilbert space H_1 equipped with the scalar product

$$\langle u, v \rangle_1 = \langle u, (1 + bA)v \rangle, \tag{2.3}$$

where b is a positive real number. The norm determined by the latter scalar product is equivalent to the standard norm in the space H_1 and is given by

$$\| u \|_1^2 = \langle u, (1 + bA)u \rangle.$$

Then the domain of the operator A coincides with the space H_3 and the operator is self–adjoint on this domain. The operator A^0 being the restriction of the operator A to the domain

$$\mathrm{Dom}(A^0) = \{ u \in H_3 : \langle \varphi, u \rangle = 0 \} \tag{2.4}$$

is a densely defined symmetric operator, since φ satisfies (2.1). The domain of the self–adjoint operator corresponding to the formal expression (1.4) necessarily contains the element $g_1 = \frac{1}{A + a_1} \varphi \in H_{-1}$. Therefore the extension of the operator

A^0 corresponding to formal expression (1.4) cannot be constructed in the Hilbert space H_1. Let us consider the one dimensional extension $\mathcal{H}$ of this space,

$$\mathcal{H} = \mathrm{Dom}(A^0)\dot{+}\mathbf{C} \ni \mathcal{U} = (u, u_1). \tag{2.5}$$

Note that $\mathcal{H} \subset H_3\dot{+}\mathbf{C}$. We define the following natural embedding ρ of the space $\mathcal{H}$ into the space H_{-1}:

$$\begin{array}{rcl} \rho & : & \mathcal{H} \to H_{-1} \\ & & (u, u_1) \mapsto u + u_1 g_1. \end{array} \tag{2.6}$$

We define a sesquilinear form on the domain $\mathrm{Dom}(A^0)\dot{+}\mathbf{C}$ as follows:

(2.7)

$$\ll \mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} \; = \; \langle u, v\rangle + b\langle u, Av\rangle + d\bar{u}_1 v_1 + (1 - ba_1)\left\{\bar{u}_1\langle g_1, v\rangle + v_1\langle u, g_1\rangle\right\}.$$

This form defines a scalar product only if it is positive definite.

Let us denote by $\| \mathcal{U} \|_{\mathcal{H}}^2 = \ll \mathcal{U}, \mathcal{U} \gg_{\mathcal{H}}$ the norm associated with the previously introduced scalar product. The space $\mathcal{H}$ with this norm is not complete, and the following lemma describes its completion with respect to this norm.

Lemma 2.1. *Let the following inequality be satisfied*

$$d > |1 - ba_1|^2 \, \| g_1 \|_{-1}^2 \, . \tag{2.8}$$

Then the norm $\| \cdot \|_{\mathcal{H}}$ *is equivalent to the standard norm in the Hilbert space* $H_1 \oplus \mathbf{C}$

$$\| \mathcal{U} \|^2 \equiv \| (u, u_1) \|^2 = \langle u, (1 + bA)u\rangle + |u_1|^2.$$

The completion of the space $\mathcal{H} = \mathrm{Dom}(A^0)\dot{+}\mathbf{C}$ *with respect to the norm* $\| \cdot \|_{\mathcal{H}}$ *coincides with the space* $H_1\dot{+}\mathbf{C}$.

Note that the scalar product in the space $\overline{\mathcal{H}}$ calculated on the vectors with the component u_1 equal to zero is equivalent to the scalar product in the space H_1. But the decomposition of the space $\overline{\mathcal{H}} = H_1\dot{+}\mathbf{C}$ is not orthogonal. Only if $ba_1 = 1$ does the decomposition become orthogonal. We are going to use the same notation $\mathcal{H}$ for the completed space.

3. The Operator

We define the operator $\mathcal{A}$ on the set of regular elements $\mathrm{Dom}_r \subset \mathcal{H}$ which possess the representation

$$\mathcal{U} = (u, u_1) = (u_r + u_2 g_2, u_1),$$

where $u_r \in H_3, u_2 \in \mathbf{C}$. The vector

$$g_2 = \frac{1}{A + a_2} g_1 = \frac{1}{A + a_2}\frac{1}{A + a_1}\varphi \in H_1$$

is defined using another one positive parameter, $a_2 > 0$. The embedding operator ρ maps every such element to a vector from H_{-1} as follows:

$$\rho(u_r + u_2 g_2, u_1) = u_r + u_2 g_2 + u_1 g_1.$$

Then the operator $\mathcal{A}$ in $\mathcal{H}$ is defined on Dom_r in such a way that the following equality holds:

$$A\rho\mathcal{U} \equiv \rho\mathcal{A}\mathcal{U} (\mathrm{mod}\varphi), \tag{3.1}$$

where this equality in H_{-3} holds if and only if the difference between the left and right hand sides is proportional to $\varphi \in H_{-3}$. In other words, the operator $\mathcal{A}$ acts as formal adjoint operator. There exists a unique operator $\mathcal{A}$ in $\mathcal{H}$ satisfying (3.1)

$$\mathcal{A}\mathcal{U} = \mathcal{A}\begin{pmatrix} u_r + u_2 g_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} Au_r - a_2 u_2 g_2 \\ u_2 - a_1 u_1 \end{pmatrix}. \tag{3.2}$$

The operator $\mathcal{A}$ is not self–adjoint. In fact it is not even symmetric. The boundary form of the operator can be calculated explicitly. We present here the result of these tedious but otherwise straightforward calculations

$$\begin{aligned} \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} - \ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg_{\mathcal{H}} = < u_r, \varphi > (bv_2 + (1 - ba_1)v_1) \\ - (b\bar{u}_2 + (1 - ba_1)\bar{u}_1) < \varphi, v_r > + a\{\bar{u}_2 v_1 - \bar{u}_1 v_2\}, \end{aligned} \tag{3.3}$$

where we put $a = d + (ba_1 - 1) < g_1, g_2 > (a_2 - a_1)$. This formula defines the following sesquilinear form in $\mathbf{C}^3$:

$$\ll \mathcal{U}, \mathcal{A}\mathcal{V} \gg_{\mathcal{H}} - \ll \mathcal{A}\mathcal{U}, \mathcal{V} \gg_{\mathcal{H}} = \tag{3.4}$$

$$= \left\langle \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} < \varphi, u_r > \\ u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} < \varphi, v_r > \\ v_1 \\ v_2 \end{pmatrix} \right\rangle,$$

where $c = ba_1 - 1 \in \mathbf{R}$. The rank of the 3×3 matrix appeared in this formula is equal to 2 if at least one of the parameters a, b, c is different from zero, since the characteristic polynomial for the matrix is given by $-\lambda(\lambda^2 + a^2 + b^2 + c^2)$. Then all symmetric restrictions of the operator $\mathcal{A}$ can be defined by certain boundary conditions imposed on the functions from the domain of the operator. The problem of defining a symmetric restriction of $\mathcal{A}$ is equivalent to the problem of finding a Lagrangian plane of the boundary form.

Suppose that the boundary conditions are written in the form

$$\alpha < \varphi, u_r > + \beta u_1 + \gamma u_2 = 0, \tag{3.5}$$

where $\alpha, \beta, \gamma \in \mathbf{C}$ are certain complex parameters, not all equal to zero simultaneously. And suppose that the boundary form is equal to zero if $\mathcal{U}, \mathcal{V}$ satisfy (3.5). Then we can verify that the complex parameters α, β and γ have equal phase. Hence, without loss of generality, we can restrict our consideration to the case of real parameters, since the boundary condition (3.5) is linear. The condition can be written as

$$\alpha a + \beta b + \gamma c = 0 \Leftrightarrow (\alpha, \beta, \gamma) \perp (a, b, c). \tag{3.6}$$

The symmetric restrictions of $\mathcal{A}$ have been described by three real parameters $(\alpha, \beta, \gamma) \in \mathbf{R}^3$ satisfying (3.6). Since the length of the vector (α, β, γ) does not play any role, all of the Lagrangian planes can be parameterized by one real parameter $\theta \in [0, 2\pi)$ as follows:

$$(\alpha, \beta, \gamma) = (b \sin\theta, -a \sin\theta - c\cos\theta, b\cos\theta), \tag{3.7}$$

where we have taken into account that b is not equal to zero. We are going to use the following definition in what follows

DEFINITION 3.1. The operator $\mathcal{A}_\theta$, $\theta \in [0, 2\pi)$, is the restriction of the operator $\mathcal{A}$ defined by (3.2) to the domain of functions $\mathcal{U} = (u, u_1) \in \mathcal{H}$ possessing the representation

$$(u, u_1) = (u_r + u_2 g_2, u_1), \ u_r \in H_3, u_{1,2} \in \mathbf{C}$$

and satisfying the boundary condition

$$b\sin\theta\langle\varphi, u_r\rangle - (a\sin\theta + c\cos\theta)u_1 + b\cos\theta u_2 = 0. \tag{3.8}$$

THEOREM 3.2. *The operator $\mathcal{A}_\theta$ is a self–adjoint operator in $\mathcal{H}$ with the scalar product $\ll\cdot,\cdot\gg_{\mathcal{H}}$.*

We can prove that the operator $\mathcal{A}_\theta$ is not only self–adjoint, but semibounded from below. Moreover the resolvent of the operator can be calculated. The solution $\mathcal{U}$ of $(\mathcal{A}_\theta - \lambda)\mathcal{U} = \mathcal{V}$ is given by

$$\begin{aligned}
\langle\varphi, u_r\rangle &= \frac{(a\sin\theta+(c-b(a_1+\lambda))\cos\theta)\langle\varphi,\frac{1}{A-\lambda}v\rangle+(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle(a\sin\theta+c\cos\theta)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)};\\
u_1 &= \frac{b\sin\theta\langle\varphi,\frac{1}{A-\lambda}v\rangle+b\big(\cos\theta+(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\sin\theta\big)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)};\\
u_2 &= \frac{b\sin\theta(a_1+\lambda)\langle\varphi,\frac{1}{A-\lambda}v\rangle+(a\sin\theta+c\cos\theta)v_1}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)}.
\end{aligned}$$

Then the resolvent can be calculated as

$$\frac{1}{\mathcal{A}_\theta-\lambda}(v, v_1) = \left(\frac{1}{A-\lambda}v + \left(\frac{1}{A-\lambda}g_1\right)u_2, u_1\right). \tag{3.9}$$

The resolvent restricted to the subspace $H_1 \subset \mathcal{H}$ of functions $\mathcal{V} = (v, v_1) \in \mathcal{H}$ with zero component $v_1 = 0$ is given by

$$\begin{aligned}
\rho\frac{1}{\mathcal{A}_\theta-\lambda}|_{H_1}v &= \frac{1}{A-\lambda}v + \\
&+\frac{b\sin\theta}{\cos\theta(c-b(a_1+\lambda))+\sin\theta\big(a-b(a_1+\lambda)(a_2+\lambda)\langle\varphi,\frac{1}{A-\lambda}g_2\rangle\big)}\left(\frac{1}{A-\lambda}\varphi\right)\left\langle\varphi,\frac{1}{A-\lambda}v\right\rangle.
\end{aligned} \tag{3.10}$$

Consider the special case $\theta = 0$. In this case the resolvent and the restricted resolvent are given by

$$\frac{1}{\mathcal{A}_0-\lambda}\mathcal{V} = \left(\frac{1}{A-\lambda}v + \left(\frac{1}{A-\lambda}g_1\right)\frac{1}{\frac{c}{b}-a_1-\lambda}v_1, \frac{1}{\frac{c}{b}-a_1-\lambda}v_1\right) \tag{3.11}$$

and

$$\frac{1}{A-\lambda}|_{H_1} = \frac{1}{\mathcal{A}_0-\lambda}|_{H_1}, \tag{3.12}$$

respectively. The range of the restricted resolvent in this case is a subset of H_1 again. Moreover the restricted resolvent coincides with the resolvent of the original operator A, and this property is characteristic of the operator $\mathcal{A}_0$. In other words, the domain of the operator $\mathcal{A}_0$ contains the domain of the original operator A, and the action of the operators $\mathcal{A}_0$ and A restricted to this domain coincide,

$$\mathcal{A}_0|_{\mathrm{Dom}(A)} = A.$$

Therefore the operator $\mathcal{A}_0$ should be considered as an **unperturbed operator**, since this is the unique operator possessing the properties described above. All of the other operators $\mathcal{A}_\theta$ corresponding to $\theta \neq 0$ are perturbations of $\mathcal{A}_0$.

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