Interface Regularity for Maxwell and Stokes Systems

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Abstract

Interface regularity for stationary Maxwell, Stokes, and Navier-Stokes systems is studied. We show that the localized interface regularity holds in the normal direction of the solution, under least assumptions on the continuity of the solution.

1 Introduction

I would like to talk about interface regularity of three dimensional Maxwell and Stokes systems. To our knowledge, not so much regards have been taken in this topic, but actually the solenoidal condition provides the regularity across interface to a specified component of the unknown function.

Let $\Omega \subset \mathbf{R}^3$ and $\mathcal{M} \subset \mathbf{R}^3$ be a bounded domain and a C^2 hypersurface, respectively, and suppose that $\mathcal{M}$ intersects with $\partial\Omega$ transversally, where $\partial\Omega$ denotes the boundary of Ω. That is,

$$\begin{aligned} &\mathcal{M} \cap \Omega \neq \phi, \\ &\Omega = \Omega_+ \cup (\Omega \cap \mathcal{M}) \cup \Omega_- \quad \text{(disjointunion)}, \end{aligned} \tag{1}$$

with $\Omega_\pm$ being open subsets of Ω. First, we take the Maxwell system in magnetostatics,

$$\left.\begin{aligned} \nabla \times B = J \\ \nabla \cdot B = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_\pm, \tag{2}$$

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where $B=(B^1(x),B^2(x),B^3(x))$ and $J=(J^1(x),J^2(x),J^3(x))$ stand for the three dimensional vector fields, indicating magnetic field and total current density, respectively. Here and henceforth, $\nabla = {}^T(\partial_1,\partial_2,\partial_3)$ denotes the gradient operator and $\times$ and $\cdot$ are the outer and the inner products in $\mathbf{R}^3$, so that $\nabla\times$ and $\nabla\cdot$ are the rotation and the divergence operations, respectively. We put $[A]_-^+ = A_+ - A_-$ where

$$A_+(\xi) = \lim_{x\to\xi,\, x\in\mathbf{R}^3\setminus D} A(x), \qquad A_-(\xi) = \lim_{x\to\xi,\, x\in D} A(x)$$

for $\xi\in\partial D$.

To state the result, we take preliminaries from Girault and Raviart [1]. Let $D\subset\mathbf{R}^3$ be a locally Lipschitz domain $D\subset\mathbf{R}^3$ with the boundary $\Gamma=\partial D$ and the unit normal vector n to Γ. The Sobolev space $H^m(D)$ is defined by

$$H^m(D) = \left\{ u\in L^2(D) \mid \partial^\alpha u\in L^2(D) \quad \text{for} \quad |\alpha|\le m \right\},$$

if m is a positive integer, where $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for the multi-index $\alpha = (\alpha_1,\alpha_2,\alpha_3)$. Given $\sigma\in(0,1)$, we say that $u\in H^{m+\sigma}(D)$ if $u\in H^m(D)$ and

$$\int_D\int_D \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dxdy < +\infty$$

for any α in $|\alpha|=m$ and $n=3$. The space $H^s(\Gamma)$ for $s\in[0,1]$ is defined similarly with $n=2$ through the local chart of Γ, and we set $H^{-s}(\Gamma) = H^s(\Gamma)'$. Finally, we put

$$H(\mathrm{div}, D) = \left\{ u\in L^2(D)^3 \mid \nabla\cdot u\in L^2(D) \right\}$$

and

$$H(\mathrm{rot}, D) = \left\{ u\in L^2(D)^3 \mid \nabla\times u\in L^2(D)^3 \right\}.$$

Then, we have $n\cdot v|_\Gamma \in H^{-1/2}(\Gamma)$ and the Green's formula

$$((v,\nabla\varphi))_D + (\nabla\cdot v,\varphi)_D = \langle n\cdot v,\varphi\rangle_\Gamma$$

holds for $v\in H(\mathrm{div},D)$ and $\varphi\in H^1(D)$, where $(\cdot,\cdot)_D$ and $((\cdot,\cdot))$ denote $L^2(D)$ and $L^2(D)^3$ inner products, respectively, and $\langle\cdot,\cdot\rangle_\Gamma$ the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Similarly, we have $n\times v|_\Gamma\in H^{-1/2}(\Gamma)^3$ and the Stokes formula

$$((\nabla\times v,w))_D - ((v,\nabla\times w))_D = \langle\langle n\times v,w\rangle\rangle_\Gamma$$

holds for $v \in H^1(D)^3$ and $w \in H^1(D)^3$, where $\langle\langle \cdot, \cdot \rangle\rangle_\Gamma$ denotes the duality pairing between $H^{-1/2}(\Gamma)^3$ and $H^{1/2}(\Gamma)^3$.

Recalling the Maxwell system (2), now we discuss the interface regularity of the solution B. Put

$$\Gamma_\pm = \partial\Omega_\pm \cap M$$

with $\partial\Omega_\pm$ being the boundary of $\Omega_\pm$. Thus, Γ_+ and Γ_- coincide as sets, but they are regarded as parts of the boundaries of Ω_+ and Ω_-, respectively. Henceforth, n denotes the outer unit normal vector to Γ_-. Therefore, $-n$ is nothing but the outer unit normal vector to Γ_+. Henceforth, a C^2 extension to Ω of the vector field n on $\Gamma \equiv \mathcal{M} \cap \Omega$ is taken. Furthermore, given a function $A(x)$ on $\Omega_\pm$, we set

$$[A]_-^+ = A_+ - A_- \qquad \text{on} \qquad \Gamma,$$

where $A_\pm(\xi) = \lim_{x\to\xi,\, x\in\Omega_\pm} A(x)$ for $\xi \in \Gamma$.

Suppose that B and J are in $L^2(\Omega_\pm)^3$ and satisfy (2). Then it holds that $B \in H(\mathrm{rot}, \Omega_\pm) \cap H(\mathrm{div}, \Omega_\pm)$. This implies

$$n \times B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \qquad \text{and} \qquad n \cdot B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm),$$

and hence $B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$. Furthermore, $[n \times B]_-^+ = 0$ and $[n \cdot B]_-^+ = 0$ on Γ if and only if $\nabla \times B \in L^2(\Omega)^3$ and $\nabla \cdot B \in L^2(\Omega)$ as distributions, respectively. In particular, $B \in H^1_{loc}(\Omega)^3$ if and only if $[B]_-^+ = 0$ on Γ by Corollary I.2.10 of [1], that is, $H(\mathrm{rot}, \Omega) \cap H(\mathrm{div}, \Omega) \subset H^1_{loc}(\Omega)^3$.

Under those preparations, our first result is stated as follows. Note again that $n \cdot B$ is extended to a function in Ω.

Theorem 1 *If $B \in H^1(\Omega)^3$ and $J \in H(rot, \Omega_\pm)$ satisfy (2), then it holds that $n \cdot B \in H^2_{loc}(\Omega)$.*

In the above theorem, B solves (2) in Ω because it is in $H^1_{loc}(\Omega)^3$. On the other hand, $J \in H(\mathrm{rot}, \Omega_\pm)$ belongs to $J \in H(\mathrm{rot}, \Omega)$ if and only if $[n \times J]_-^+ = 0$ holds. If this condition is satisfied, then it holds that

$$-\Delta B = \nabla \times J \in L^2(\Omega)^3$$

as a distribution by $\nabla \cdot B = 0$ in Ω. This implies $B \in H^2_{loc}(\Omega)^3$ from the elliptic regularity. Thus, Theorem 1 says that even when $n \times J$ has an interface on $\Gamma = \mathcal{M} \cap \Omega$, $n \cdot B$ gains the regularity in one rank.

Theorem 1 can be applied to the stationary Stokes system;

$$\left.\begin{array}{r} -\Delta v + \nabla p = f \\ \nabla \cdot v = 0 \end{array}\right\} \qquad \text{in} \qquad \Omega_{\pm} \tag{3}$$

and the stationary Navier-Stokes system;

$$\left.\begin{array}{r} -\Delta v + (v \cdot \nabla)\, v + \nabla p = f \\ \nabla \cdot v = 0 \end{array}\right\} \qquad \text{in} \qquad \Omega_{\pm}, \tag{4}$$

where $v = (v^1(x), v^2(x), v^3(x))$ denotes the velocity of fluid, $p = p(x)$ the pressure, and $f(x) = (f^1(x), f^2(x), f^3(x))$ the external force. Any components of those vector or scalar fields are supposed to be in $L^1_{loc}(\Omega)$. We have the following theorem, where $\omega = \nabla \times v$ indicates the vorticity of fluid.

Theorem 2 *If $v \in H^2(\Omega_{\pm})^3$, $\nabla p \in L^2(\Omega_{\pm})^3$, and $f \in H(rot, \Omega_{\pm})$ satisfy (3) or (4) and $\omega = \nabla \times v$ is in $H^1(\Omega)^3$, then it holds that $n \cdot \omega \in H^2_{loc}(\Omega)$.*

In the above theorem, systems of equations are supposed to hold piecewisely in Ω, and np and $n \times f$ may have interfaces. Neverthless, the normal component of the vorticity ω gains the regularity in one rank if $[\omega]^+_- = 0$ holds on $\Gamma = \Omega \cap \mathcal{M}$. On the other hand, the following theorem corresponds to Theorem 1, and there, the equations in system (3) or (4) hold in Ω.

Theorem 3 *If $\mathcal{M} \subset \mathbf{R}^3$ is C^3 and $v \in H^2(\Omega)^3$, $\nabla p \in H^1(\Omega_{\pm})^3$, and $f \in H^1(\Omega_{\pm})^3$ satisfy (3) with $\frac{\partial f}{\partial x_j} \in H(rot, \Omega_{\pm})^3$ for $j = 1, 2, 3$, then it holds that $n \cdot v \in H^3_{loc}(\Omega)$. The same conclusion follows for (4) if $\omega = \nabla \times v \in (H^2(\Omega_{\pm}))^3$ is imposed besides other conditions.*

2 Key Lemma and Proof of Theorem 1

In this section, we consider the Maxwell system (2) and discuss the interface regularity of the solution B. First, we show the following.

Lemma 2.1 *If $B \in L^2(\Omega_{\pm})^3$ and $J \in H(rot, \Omega_{\pm})$ satisfy (2), then it follows that*

$$\begin{aligned} &\langle\langle \nabla(n \cdot B), C \rangle\rangle^+_- - \langle\langle (\nabla \cdot n) B, C \rangle\rangle^+_- \\ &\quad = \langle\langle B, (n \cdot \nabla) C \rangle\rangle^+_- - \langle\langle n \times B, \nabla \times C \rangle\rangle^+_- - \langle n \cdot B, \nabla \cdot C \rangle^+_- \end{aligned} \tag{5}$$

for any $C \in C_0^\infty(\Omega)^3$, where $\langle\langle\ ,\ \rangle\rangle^+_- = \langle\langle\ ,\ \rangle\rangle_{\Gamma_+} - \langle\langle\ ,\ \rangle\rangle_{\Gamma_-}$.

Proof of Theorem 1: Since $[B]_-^+ = 0$ on Γ, we have

$$\langle\langle \nabla(n \cdot B), C \rangle\rangle_-^+ = 0$$

for any $C \in \{C_0^\infty(\Omega)\}^3$. This implies

$$[\nabla(n \cdot B)]_-^+ = 0 \quad \text{on} \quad \Gamma. \tag{6}$$

We have $B \in H^1_{loc}(\Omega)^3$ and $-\Delta B = \nabla \times J \in L^2(\Omega_\pm)$. Hence $\Delta(n \cdot B) \in L^2(\Omega_\pm)$ follows. Because $\nabla(n \cdot B)|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ satisfies (6), the Green's formula now gives

$$\int_\Omega \Delta(n \cdot B)\psi dx = \int_\Omega (n \cdot B)\Delta\psi dx$$

for any $\psi \in C_0^\infty(\Omega)$. This means that if $f \in L^2(\Omega)$ is defined by

$$f = \begin{cases} \Delta\,(n \cdot B)|_{\Omega_+} & \text{in} \quad \Omega_+ \\ \Delta\,(n \cdot B)|_{\Omega_-} & \text{in} \quad \Omega_-, \end{cases}$$

then it follows that $\Delta(n \cdot B) = f \in L^2(\Omega)$ as a distribution in Ω. In use of $n \cdot B \in H^1(\Omega)$ and the elliptic regularity, we obtain $n \cdot B \in H^2_{loc}(\Omega)$. The proof is complete.

In the rest of this section, we take the case that $\mathcal{M}$ is flat. If $B \in H^1(\Omega_\pm)^3$, then $B|_{\Gamma_\pm} \in H^{1/2}(\Gamma_\pm)^3$ and hence $(n \times \nabla)\,B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ is well-defined. If $\nabla \cdot B = 0$ in $\Omega_\pm$ furthermore, then

$$(n \cdot \nabla)\,(n \cdot B)|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)$$

follows from those relations as we shall see. In this case, Theorem 1 has component-wise version.

Theorem 4 *Suppose that the interface $\mathcal{M}$ is flat, and that $B \in H^1(\Omega_\pm)^3$ and $J \in H(rot, \Omega_\pm)$ satisfy (2). Then, if $[n \cdot B]_-^+ = 0$ on Γ it holds that $[(n \times \nabla)(n \cdot B)]_-^+ = 0$ on Γ. Similarly, if $[n \times B]_-^+ = 0$ on Γ we have $[(n \cdot \nabla)(n \cdot B)]_-^+ = 0$ on Γ.*

3 Stokes and Navier-Stokes system

Theorem 5 *Assume that system* (3) *holds with* $v \in H^2(\Omega_\pm)^3$, $\nabla p \in L^2(\Omega_\pm)^3$, $f \in H^1(\Omega_\pm) \cap H(\mathrm{rot}, \Omega_\pm)$ *and* $(n \cdot \nabla) f \in H(\mathrm{rot}, \Omega_\pm)$. *If the conditions* $[n \times v]_-^+ = 0$ *and* $[(n \cdot \nabla)(n \cdot v)]_-^+ = [n \cdot (\nabla \times v)]_-^+ = 0$ *are satisfied on* Γ, *then it holds that* $n \times v|_\Gamma \in H^{5/2}_{loc}(\Gamma)^3$.

Theorem 6 *Under the assumptions of Theorem 5, if the conditions* $[(n \cdot \nabla)(n \times v)]_-^+ = [n \times (n \cdot \nabla) v]_-^+ = [n \times (\nabla \times v)]_-^+ = 0$ *are satisfied on* Γ, *then it holds that* $(n \cdot \nabla)(n \times v)|_\Gamma \in H^{3/2}_{loc}(\Gamma)^3$.

Theorem 7 *The same conclusions as in Theorems 5 and 6 hold to system (4) if* $\omega = \nabla \times v \in L^\infty(\Omega_\pm)^3$ *and* $n \cdot \nabla \omega \in H^1(\Omega_\pm)^3$ *are imposed besides other conditions.*

References

[1] V. Girault and P-A. Raviart "Finite element methods for Navier-Stokes equations." Springer-Verlag, Berlin, 1986.

[2] T. Suzuki, K. Watanabe and M. Shimogawara, *Current status and mathematical analysis for magnetoencephalography* (in Japanese), Osaka Univ. Research Reorts in Math. no.1 (2000)