GLOBAL SOLVABILITY OF THE COMPLEX GINZBURG-LANDAU EQUATION WITH DISTRIBUTION-VALUED INITIAL DATA

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1. Introduction

We consider initial value problems for the complex Ginzburg-Landau equation over 1-dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$:

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\dfrac{\partial^2 u}{\partial x^2} + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0, \quad (x,t) \in \mathbb{T} \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in \mathbb{T}, \end{cases}$$

where $q \geq 2, \lambda > 0, \kappa > 0, \alpha, \beta, \gamma \in \mathbb{R}$ are constants, $i = \sqrt{-1}$, and u is a complex-valued unknown function. $\mathbb{T}$ is usually regarded as the interval $I := [-\pi, \pi]$, identifying $-\pi$ and π. Then the initial value problem (CGL) is equivalent to the following initial-boundary value problem with periodic boundary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\dfrac{\partial^2 u}{\partial x^2} + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0, \quad (x,t) \in I \times \mathbb{R}_+, \\ u(-\pi, t) = u(\pi, t), \quad t \in \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad x \in I. \end{cases}$$

The existence of unique **global strong** solutions to (CGL) is so far known only for initial values $u_0 \in L^2(\mathbb{T})$ (cf. [**2**]). The purpose of this talk is to announce the strong solvability of (CGL) for wider class of initial values u_0 in negative order Sobolev spaces $H^s(\mathbb{T})$ $(s < 0)$. That is, we consider distribution-valued initial data. Thus, the result means smoothing effect of the solution operator.

Our result is stated as follows:

Theorem 1.1. *For $2 \leq q < 6$ let $\kappa + i\beta$ be a complex number satisfying $\dfrac{|\beta|}{\kappa} \leq \dfrac{2\sqrt{q-1}}{q-2}$. Assume that $s < 0$ satisfies the constraint:*

$$\text{(1.1)} \qquad s > \begin{cases} -\dfrac{2}{q-1} & \text{if } 2 \leq q \leq 3, \\ \dfrac{1}{2} - \dfrac{3}{q-1} & \text{if } 3 \leq q \leq 4, \\ \dfrac{1}{2} - \dfrac{2}{q-2} & \text{if } 4 \leq q < 6. \end{cases}$$

Then for every $u_0 \in H^s(\mathbb{T})$ there exists a unique global strong solution $u(\cdot)$ to (CGL).

Here $u(\cdot)$ is called the **global strong solution** if

$$u(\cdot) \in C([0,\infty); H^s(\mathbb{T})) \cap C^{0,1}_{\mathrm{loc}}((0,\infty); L^2(\mathbb{T}))$$

and furthermore, for a given $T > 0$, $u(\cdot)$ satisfies the equation in (CGL) in $L^2(\delta, T; L^2(\mathbb{T}))$ for any $\delta \in (0, T)$.

To prove the Theorem we proceed as follows. First we construct the local solution $u_{\mathrm{loc}}(\cdot)$ to (CGL) with $u_0 \in H^s(\mathbb{T})$. The solution can be chosen so that $u_{\mathrm{loc}}(t) \in L^p(\mathbb{T})$ ($p \geq 2$) a.a. t over the interval of its existence. This process has been established by Levermore-Oliver [**1**].

Next, we take $u_{\mathrm{loc}}(\delta) \in L^2(\mathbb{T})$ as a new initial value for sufficiently small $\delta > 0$ and apply the result of Okazawa-Yokota [**2**] to extend the solution globally.

2. Local Existence (Contraction Method)

We describe the result of [**1**] about the construction of local solutions to (CGL) by Contraction Methods. Since $\lambda > 0$, the opeator $(\lambda + i\alpha)(d/dx)^2 + \gamma$ is the infinitesimal generator of the C_0-semigroup

$$U(t) := e^{\gamma t} \exp((\lambda + i\alpha)t(d/dx)^2)$$

on $L^2(I)$ as well as on $H^s(I)$. Then (CGL) can be formally recast as an integral equation (mild formulation):

$$\text{(IE)} \qquad u(t) = U(t)u_0 - (\kappa + i\beta) \int_0^t U(t-s)|u|^{q-2}u(s)\, ds.$$

Here, for a given $v_0 \in L^2(I)$ it is well known that $\{U(t); t \geq 0\}$ is given by the convolution

$$U(t)v_0 = H_t * v_0, \quad v_0 \in L^2(I),$$

where the integral kernel $H_t = H_t(x)$ is given by

$$H_t(x) := \sum_{n \in 2\pi\mathbb{Z}} h_t(x+n), \quad x \in I,$$

$$h_t(x) := \frac{e^{\gamma t}}{(4\pi(\lambda + i\alpha)t)^{\frac{1}{2}}} \exp\Big(-\frac{x^2}{4(\lambda + i\alpha)t}\Big), \quad x \in \mathbb{R}$$

(cf. Taylor [**3**, Section 3.7] in which $\lambda = 1$, $\alpha = 0$). Note that solutions of (IE) are mild solutions of (CGL).

The following local existence theorem is due to Levermore-Oliver [**1**].

Theorem 2.1. *For $2 \leq q < 6$ let $s < 0$ satisfy the constraint* (1.1) *in* Theorem 1.1. *Then for every $\rho > 0$ there exists a time $T = T(\rho) > 0$ and $2 \leq p < 10$ such that for every initial value $u_0 \in H^s(\mathbb{T})$ with $\|u_0\|_{H^s} \leq \rho$,* (IE) *admits a unique solution*

$$u(\cdot) \in C([0,T]; H^s(\mathbb{T})) \cap E^\theta((0,T]; L^p(\mathbb{T})),$$

where $\theta := \frac{1}{2}\Big(\frac{1}{2} - \frac{1}{p} - s\Big)$, and $E^\theta((0,T]; L^p(\mathbb{T}))$ is given by

$$E^\theta((0,T]; L^p(\mathbb{T})) := \{w(\cdot) \in C((0,T]; L^p(\mathbb{T})); \sup\{t^\theta \|w(t)\|_{L^p}; 0 < t \leq T\} < \infty\}$$

with norm

$$\|w(t)\|_{E^\theta} := \sup\{t^\theta \|w(t)\|_{L^p}; 0 < t \leq T\}.$$

To prove this theorem, we introduce the following operator

$$Nu := (\kappa + i\beta)|u|^{q-2}u \ \text{ for } \ u \in D(N) := L^{2(q-1)}(\mathbb{T}).$$

Then (IE) is recast as an abstract integral equation:

$$\text{(AIE)} \qquad u(t) = U(t)u_0 - \int_0^t U(t-s)(Nu)(s)\,ds.$$

Under the above settings, the following abstract theorem holds (see [**1**, Theorem 3]).

Theorem 2.2 (General Local Existence). *Let $\{U(t); t \geq 0\}$ be a C_0-semigroup on a Banach space X. Let Y be another Banach space continuously embedded in X and $N : Y \to X$ a nonlinear operator. Assume that $\{U(t)\}$ and N satisfy the following two conditions*:

(I) $U(t)X \subset Y \ \forall\, t > 0$ *and there exist two constants* θ $(0 < \theta < 1)$ *and* M $(M > 0)$ *such that*

$$\|U(t)w\|_Y \leq Mt^{-\theta}\|w\|_X \ \ \forall\, w \in X.$$

(II) *There exist two constants $q \geq 2$ and $L > 0$ such that*

$$\|Nu - Nv\|_X \leq L(\|u\|_Y + \|v\|_Y)^{q-2}\|u - v\|_Y \ \ \forall\, u, v \in Y.$$

If $q < 1 + 1/\theta$, then for every $\rho > 0$ there exists $T = T(\rho) > 0$ such that for every initial value $u_0 \in X$ with $\|u_0\|_X \leq \rho$ the integral equation (AIE) *admits a unique solution*

$$u(\cdot) \in C([0,T]; X) \cap E^\theta((0,T]; Y),$$

where $E^\theta((0,T]; Y)$ is given by

$$E^\theta((0,T]; Y) := \{w(\cdot) \in C((0,T]; Y); \sup\{t^\theta\|w(t)\|_Y; 0 < t \leq T\} < \infty\}$$

with norm

$$\|w(t)\|_{E^\theta} := \sup\{t^\theta\|w(t)\|_Y; 0 < t \leq T\}.$$

By setting $X := H^s(\mathbb{T})$, $Y := L^p(\mathbb{T})$ $(p \geq 2)$ in Theorem 2.2, we can obtain Theorem 2.1.

3. Global Existence (Monotonicity Method)

We describe the result in [**2**] about the construction of the global solution by Monotonicity Methods.

First we formulate the problem (CGL) in the complex Hilbert space $X := L^2(\mathbb{T})$. To this end we introduce two m-accretive operators:

$$\begin{aligned}
&D(S) := H^2(\mathbb{T}) = \{u \in H^2(I); u(-\pi) = u(\pi), u'(-\pi) = u'(\pi)\},\\
&(Su)(x) := -(d/dx)^2u(x) \ \text{ for } \ u \in D(S),\\
&(Bu)(x) := |u|^{q-2}u(x) \ \text{ for } \ u \in D(B) := L^{2(q-1)}(\mathbb{T}).
\end{aligned}$$

Then (CGL) is regarded as the following abstract nonlinear evolution equation:

$$\text{(DE)} \qquad \begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)Bu - \gamma u = 0, \quad t > 0,\\ u(0) = u_0 \in X. \end{cases}$$

The following global existence theorem holds (see [**2**, Theorem 1.3])

Theorem 3.1. *Let $\kappa + i\beta$ be a complex number satisfying $\kappa^{-1}|\beta| \le (2\sqrt{q-1})/(q-2)$. Then for any $u_0 \in L^2(\mathbb{T})$ there exists a "unique" strong solution to* (CGL).

The proof of Theorem 3.1 is completed by regarding B as a subdifferential operator in $L^2(\mathbb{T})$.

Let S be a nonnegative selfadjoint operator in a complex Hilbert space X and $S^{1/2}$ its square root. Let ψ be a proper lower semi-continuous convex function on X. For simplicity we assume that $\psi \ge 0$ and $\partial\psi$ is single-valued. Now we consider the abstract Cauchy problem in X:

$$\text{(ACP)} \qquad \begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0. \end{cases}$$

For the convex function ψ and its subdifferential $\partial\psi$ assume that the following three conditions are satisfied:

(A1) $\exists\, q \in [2,\infty)$; $\psi(\zeta u) = |\zeta|^q \psi(u)$, $u \in D(\psi)$, $\mathrm{Re}\,\zeta > 0$.

(A2) $\exists\, \omega_q \in [0,\pi/2)$; for $u, v \in D(\partial\psi)$

$$|\mathrm{Im}(\partial\psi(u) - \partial\psi(v), u - v)| \le (\tan\omega_q)\mathrm{Re}(\partial\psi(u) - \partial\psi(v), u - v).$$

(A3) $|\mathrm{Im}(Su, \partial\psi_\varepsilon(u))| \le (\tan\omega_q)\mathrm{Re}(Su, \partial\psi_\varepsilon(u))\ \forall\, u \in D(S)$, where $\partial\psi_\varepsilon$ is the Yosida approximation of $\partial\psi$.

Theorem 3.2 (General Global Existence). *Let $\kappa+i\beta$ be a complex number satisfying $\kappa^{-1}|\beta| \le (\tan\omega_q)^{-1}$. Assume that conditions* (A1)–(A3) *are satisfied. Then for every $u_0 \in \overline{D(S^{1/2}) \cap D(\psi)}$,* (ACP) *admits a unique global strong solution*

$$u(\cdot) \in C([0,\infty); X) \cap C^{0,1}_{\mathrm{loc}}((0,\infty); X).$$

Put $X := L^2(\mathbb{T})$ in the above Theorem 3.2 and take

$$\psi(u) := \begin{cases} \dfrac{1}{q}\|u\|^q_{L^q} & \text{for } u \in L^q(\mathbb{T}), \\ +\infty & \text{otherwise.} \end{cases}$$

Then we have that the L^2-closure of $\overline{D(S^{1/2}) \cap D(\psi)} = H^1(\mathbb{T}) \cap L^q(\mathbb{T})$ is equal to $L^2(\mathbb{T})$ and $B = \partial\psi$. Moreover, all the conditions (A1)–(A3) are satisfied with $\tan\omega_q = (q-2)/(2\sqrt{q-1})$. Consequently, we can obtain Theorem 3.1.

References

[1] C.D. Levermore and M. Oliver, *Distribution-valued initial data for the complex Ginzburg-Landau equation*, Comm. Partial Differential Equations **22** (1997), 39–48.

[2] N. Okazawa and T. Yokota, *Global existence and smoothing effect for the complex Ginzburg-Landau equation with p-Laplacian*, J. Differential Equations **182** (2002), 541–576.

[3] M.E. Taylor, *Partial Differential Equations* I: *Basic Theory*, Applied Math. Sciences **115**, Springer-Verlag, New York and Berlin, 1996.