ON THE ASYMPTOTICS FOR CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

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We consider the Cauchy problem for the cubic nonlinear Schrödinger equation

$$\left\{ \begin{array}{l} \mathcal{L}u = \mathcal{N}(u),\ x \in \mathbf{R},\ t > 1 \\ u(1,x) = u_1(x),\ x \in \mathbf{R}, \end{array} \right. \tag{0.1}$$

where $\mathcal{L} = i\partial_t + \frac{1}{2}\partial_x^2$,

$$\mathcal{N}(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3,$$

$\lambda_j \in \mathbf{C}$, $j = 1,2,3$. For the coefficients λ_j we assume that there exists $\theta_0 > 0$ such that

$$\Re\left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\right) \geq C > 0, \tag{0.2}$$

$$\Im\left(\frac{\lambda_1}{\sqrt{3}} e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}} e^{-4ir}\right) r \geq Cr^2, \tag{0.3}$$

for all $|r| < \theta_0$, and also we suppose that the initial data $u_1(x)$ are such that

$$\left|\arg e^{-\frac{i}{2}\xi^2} \widehat{u_1}(\xi)\right| < \theta_0, \quad \inf_{|\xi| \leq 1} |\widehat{u_1}(\xi)| \geq C\varepsilon, \tag{0.4}$$

$\varepsilon > 0$ is a small constant depending on the size of the initial function in a suitable norm defined later. Note that conditions (0.2) - (0.3) can be fulfilled, for example, if we choose $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 0$ and $\theta_0 = \frac{\pi}{4}$.

We let

$$\chi(\xi) = \Re\left(\frac{\lambda_1}{\sqrt{3}} \exp(2ir_+(\xi)) - i\lambda_2 \exp(-2ir_+(\xi)) + \frac{\lambda_3}{\sqrt{3}} \exp(-4ir_+(\xi))\right) \tag{0.5}$$

and

$$\left|r_+\left(\frac{x}{t}\right)\right| \leq |\theta| + \varepsilon^{\frac{1}{4}}, \left|W_+\left(\frac{x}{t}\right)\right| \leq 2\varepsilon.$$

To state our result precisely we now give some notations. Denote the usual Lebesgue space $\mathbf{L}^p = \left\{\phi \in \mathbf{S}'; \|\phi\|_p < \infty\right\}$, where the norm $\|\phi\|_p = \left(\int_{\mathbf{R}} |\phi(x)|^p dx\right)^{1/p}$ if $1 \leq p < \infty$ and $\|\phi\|_\infty = \text{ess.sup}_{x \in \mathbf{R}} |\phi(x)|$ if $p = \infty$. For simplicity we write $\|\cdot\| = \|\cdot\|_2$. Weighted Sobolev space is

$$\mathbf{H}^{m,k} = \left\{\phi \in \mathbf{S}' : \|\phi\|_{m,k} \equiv \left\|\langle x\rangle^k \langle i\partial\rangle^m \phi\right\| < \infty\right\}, m, k \in \mathbf{R}, \langle x\rangle = \sqrt{1 + x^2}.$$

[1] This is a joint work with P.I.Naumkin

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We let $\mathcal{K} = \mathcal{F}\mathcal{U}(-t)$, where $\mathcal{U}(t)$ is the free Schrödinger group. The analytic function space $\mathbf{A}$ is defined

$$\mathbf{A} = \left\{ \phi \in \mathbf{L}^2 : \|\phi\|_{\mathbf{A}} \equiv \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left(\|\mathcal{I}^n \phi\| + \|\mathcal{K}\mathcal{I}^n \phi\|_{\infty} \right) < \infty \right\}.$$

Our main result is

Theorem 0.1. *We suppose that conditions (0.2) - (0.4) are valid with sufficiently small* $\varepsilon > 0$. *We assume that the initial data* $u_1 \in \mathbf{A}_1$. *Then there exists an* $\varepsilon_0 > 0$ *such that if* $\|u_1\|_{\mathbf{A}_1} \leq \varepsilon$, $0 < \varepsilon < \varepsilon_0$, *then there exists a unique solution* $u \in \mathbf{C}([1,\infty), \mathbf{L}^\infty)$ *of the Cauchy problem (0.1). Moreover there exist unique final state* $W_+, r_+ \in \mathbf{L}^\infty$ *and* $\gamma > 0$ *such that the following asymptotics for* $t \to \infty$

(0.6)

$$u(t,x) = \frac{(it)^{-\frac{1}{2}} W_+\left(\frac{x}{t}\right) e^{\frac{ix^2}{2t}}}{\sqrt{1 + \chi\left(\frac{x}{t}\right) \left|W_+\left(\frac{x}{t}\right)\right|^2 \log \frac{t^2}{t+x^2}}} + O\left(t^{-\frac{1}{2}} \left(1 + \log \frac{t^2}{t+x^2} \right)^{-\frac{1}{2}-\gamma} \right)$$

is valid uniformly with respect to $x \in \mathbf{R}$,$\gamma > 0$, *where* $\chi(\xi)$ *is given by (0.5).*

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