On large time behavior of solutions to the compressible Navier-Stokes equation in the half-space in $\mathbf{R}^3$

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In this talk, I am going to talk about large time behavior of solutions to the initial boundary value problem for the compressible Navier-Stokes equation on the half space of $\mathbf{R}^3$. The results in this talk were obtained in a joint work with Yoshiyuki KAGEI (Kyushu Univ.).

We consider the initial boundary value problem for the compressible Navier-Stokes equation in $\mathbf{R}^3_+ = \{x = (x', x_3); x' \in \mathbf{R}^2,\ x_3 > 0\}$:

$$
\begin{aligned}
&\rho_t + \operatorname{div} m = 0, \\
&m_t + \operatorname{div}\left(\tfrac{m\otimes m}{\rho}\right) + \nabla P(\rho) = \nu\Delta\left(\tfrac{m}{\rho}\right) + (\nu+\tilde{\nu})\nabla \operatorname{div}\left(\tfrac{m}{\rho}\right), \\
&m|_{x_3=0} = 0, \quad \rho|_{t=0} = \rho_0, \quad m|_{t=0} = m_0.
\end{aligned} \tag{1}
$$

where $\rho = \rho(t,x)$ is the density; $m = (m^1(t,x), m^2(t,x), m^3(t,x))$ the momentum; and $P = P(\rho)$ the pressure; ν and $\tilde{\nu}$ are viscosity constants satisfying $\nu > 0$ and $\frac{2}{3}\nu + \tilde{\nu} \geq 0$. (ρ_0, m_0) is the initial value, which is close to a constant state $(\rho^*, 0)$, where ρ^* is a given positive constant. We will show the following

Theorem 1. (i) *Let $u_0 = (\rho_0 - \rho^*, m_0) \in (H^3(\mathbf{R}^3_+) \times H^3(\mathbf{R}^3_+)) \cap (L^1(\mathbf{R}^3_+) \times L^1(\mathbf{R}^3_+))$ and satisfy the compatibility condition:*

$$
\begin{aligned}
&m_0|_{x_3=0} = 0, \\
&-\operatorname{div}\left(\frac{m_0\otimes m_0}{\rho_0}\right) - \nabla P(\rho_0) + \nu\Delta\left(\frac{m_0}{\rho_0}\right) + (\nu+\tilde{\nu})\nabla\operatorname{div}\left(\frac{m_0}{\rho_0}\right)\Big|_{x_3=0} = 0.
\end{aligned}
$$

Assume that $\partial_\rho P(\rho^*) > 0$ and that u_0 is sufficiently small in $H^3 \times H^3$. Then there exists a unique global solution $(\rho(t), m(t))$ of problem (1) with $U(t) = (\rho(t) - \rho^*, m(t)) \in C([0,\infty), H^3 \times H^3)$; and $U(t)$ satisfies

$$\|U(t)\|_{L^2\times L^2} = O(t^{-3/4}) \quad \textit{and} \quad \|U(t)\|_{L^\infty\times L^\infty} = O(t^{-3/2})$$

as $t \to \infty$. Also,

$$\|\partial_x U(t)\|_{L^2\times L^2} = O(t^{-9/8})$$

as $t \to \infty$.

(ii) *For* $u_0 = (\overline{\rho_0}, \overline{m_0})$ *with* $\overline{\rho_0} \in H^1$ *and* $\overline{m_0} = (\overline{m_{0,1}}, \overline{m_{0,2}}, \overline{m_{0,3}}) \in L^2$ *let* $\overline{U}(t)u_0(x) = (\overline{\rho}(t,x), \overline{m}(t,x))$ *denote the solution of the linearized problem at* $(\rho^*, 0)$*:*

$$
\begin{aligned}
&\partial_t \overline{\rho} + \operatorname{div} \overline{m} = 0 \\
&\partial_t \overline{m} - \widehat{\nu} \Delta \overline{m} - (\widehat{\nu} + \widehat{\widetilde{\nu}}) \nabla \operatorname{div} \overline{m} + p_1 \nabla \overline{\rho} = 0, \\
&\overline{m}|_{x_3=0} = 0, \quad (\overline{\rho}(0,x), \overline{m}(0,x)) = u_0(x),
\end{aligned} \tag{2}
$$

where $\widehat{\nu} = \nu/\rho^*, \widehat{\widetilde{\nu}} = \widetilde{\nu}/\rho^*, p_1 = \partial_\rho P(\rho^*)$*. Then, under the same assumptions on* $(\rho_0 - \rho^*, m_0)$ *in* (i)*, we have*

$$
\|U(t) - \overline{U}(t)u_0\|_{L^2 \times L^2} = O(t^{-1})
$$

as $t \to \infty$*, where* $u_0 = (\rho_0 - \rho^*, m_0)$.

(iii) *In addition to the same assumption on* $u_0 = (\rho_0 - \rho^*, m_0)$*, if we assume that* $\int_{\mathbf{R}^3_+} (\rho_0(x) - \rho^*)\, dx \neq 0$*, then*

$$
\|U(t)u_0\|_{L^2 \times L^2} \geq C t^{-3/4}
$$

as $t \to \infty$.

Theorem 1 is proved by combining the global existence results by Matsumura and Nishida (1983) and the decay estimates for solutions to the linearized problem at $(\rho^*, 0)$.

References

[1] Y. Kagei and T. Kobayashi, *On large time behavior of solutions to the Compressible Navier-Stokes Equations in the half space in* $\mathbf{R}^3$, to appear in Arch. Rational Mech. Anal.

[2] A. Matsumura and T. Nishida, *Initial boundary value problems for the equations of motion of compressible viscous and heat-conductive fluids,* Commun. Math. Phys. 89. pp. 445-464 (1983)