Extension criterion via two-components of vorticity on strong solutions to the 3 D Navier-Stokes equations

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Abstract.

We shall show that only two components of vorticity play an essential role to determine possibility of extension of the time interval for the local strong solution to the Navier-Stokes equations. Then we shall apply our extension theorem to regularity criterion on weak solutions due to Serrin and Beirão da Veiga. Chae–Choe proved the same criterion as Beirão da Veiga only by means of the two-components of vorticity. We deal with the critical case which they excluded. Our criterion may be regarded as the generalization of the result of Beal-Kato-Majda from L^∞ to BMO.

Let us consider the Navier-Stokes equations in $\mathbb{R}^3$

$$\text{(N-S)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \text{div } u = 0 \quad \text{in } x \in \mathbb{R}^3,\ t \in (0,T), \\ u\,|_{t=0} = u_0 \end{cases}$$

Our results read as follows.

Theorem 1. *Let $u_0 \in H^s_\sigma$ for $s > 1/2$. Suppose that u is a strong solution of (N-S) in the class*

$$u \in C([0,T); H^s_\sigma) \cap C^1((0,T); H^s_\sigma) \cap C((0,T); H^{s+2}_\sigma). \tag{1}$$

If

$$\int_{\varepsilon_0}^T \|\tilde{\omega}(t)\|_{BMO} dt < \infty \quad \textit{for some } 0 < \varepsilon_0 < T,$$

then u can be continued to the strong solution in the same class as (1) for some $T' > T$. Here $\omega \equiv \mathrm{rot}\ u = (\omega_1, \omega_2, \omega_3)$ and $\tilde{\omega} \equiv (\omega_1, \omega_2, 0)$.

Theorem 2. *Let $u_0 \in L^2_\sigma$. Suppose that u is a weak solution of (N-S) on $(0,T)$ satisfying the energy inequality of the strong form:*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u(s)\|_2^2$$

for almost all $0 \le s < T$ and all t such that $s \le t \le T$. If

$$\int_0^T \|\tilde{\omega}(t)\|_{BMO} dt < \infty,$$

then for every $0 < \varepsilon < T$, u is actually a strong solution of (N-S) in the class as (1).