WEAK INTERACTION BETWEEN SOLITARY WAVES OF GKDV EQUATIONS

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1. Introduction

In this talk, we consider the large time behavior of an multi-pulse solution of the generalized KdV equation (gKdV)

$$\begin{cases} u_t + f(u)_x + u_{xxx} = 0 \quad \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \end{cases} \tag{1}$$

where $f(u) = |u|^{p-1}u/p$ ($3 \le p < 5$). I will show that if the speed of the solitary waves are sufficiently close at the initial time, the wave going ahead becomes larger and the wave going behind becomes smaller and the distance between two solitary waves becomes larger as $t \to \infty$. This gives an example of multi-pulse solution of (1), which is not included in [11].

Let φ be a positive solution of

$$\begin{aligned} &\varphi'' - \varphi + f(\varphi) = 0 \quad \text{for } y \in \mathbb{R}, \\ &\lim_{y \to \pm\infty} \varphi(y) = 0. \end{aligned} \tag{2}$$

Then the solitary wave solution of (1) is written as

$$u(x,t) = c^{\frac{1}{p-1}} \varphi(c^{\frac{1}{2}}(x - ct - \gamma)),$$

where c is a positive number and γ is a real number. The solitary wave solutions are stable in $H^1(\mathbb{R})$ if $1 < p < 5$ and unstable if $p > 5$ (see [1], [6] and [17]).

If $f(u) = u^2$ or $f(u) = \pm u^3$, the inverse scattering theory is available. It informs us that the solution of (1) with well-localized initial data resolves into a train of solitary waves moving to the right and dispersive radiation which moves to the left. In the integrable case, Maddocks and Sachs [10] proved the stability of N-soliton solutions in $H^N(\mathbb{R})$. Although the inverse scattering theory does not apply to (1.1) with more general p, this type of asymptotic resolution appears to extend to equations with more general nonlinearities.

Pereleman [13] studied the large time asymptotics of 2-pulse solutions of nonlinear Schrödinger equations in the case where the two pulses are well-separated and moves to the opposite directions with large relative velocities. In this case, the interaction of solitary waves is rather weak. Recently Martel-Merle-Tsai ([11]) have proved the asymptotic stability of multi-pulse solutions of (1) in H^1 based on the energy arguments. They deal with the case where the solitary waves are well separated and the larger solitary wave goes ahead of the smaller one.

On the other hand, Ei-Fujii-Kunihiro [5] formally analyzed the large time behavior of multi-soliton solutions, in the case where $p = 2$ and solitons are well separated and of almost the same speed. In that case, the interaction of solitary waves plays more important role. In fact, it causes replusion of solitary waves. Our aim in this talk is to give the rigorous proof of the corresponding result for $3 \le p < 5$.

Before we state our result, we introduce several notations. Let $\|\cdot\|_{1,1}$ be the norm defined by $\|v\|_{1,1} = \|\langle x\rangle^1(1-\partial^2)^{1/2}v\|_{L^2(\mathbb{R})}$ and let $H^1_a(\mathbb{R}) = \{v \in H^1_{loc}(\mathbb{R}) \mid e^{ax}v \in H^1(\mathbb{R})\}$ equipped with the norm $\|v\|_{H^1_a} = \|e^{ax}v\|_{H^1(\mathbb{R})}$.

Assuming that

$$\text{the operator } \partial_y L \text{ has no nonzero eigenvalue in } L^2(\mathbb{R}), \tag{3}$$

we have the following result:

Theorem 1. *Assume that $3 \le p < 5$ and that (3) holds. Let I be a compact subset of $(0,\infty)$ and $c_{0,i}$, $c_{0,2} \in I$. Let $d_0 = \min_{i=1,2}\sqrt{c_{0,i}}$, $0 < a_1 < a < a_2 \le d_0/100$, and let*

$$\begin{gathered}
u_0(x) = \varphi_{c_{1,0}}(x - x_{1,0}) + \varphi_{c_{2,0}}(x - x_{2,0}) + v_0(x - x_{1,0}),\\
\|v_0\|_{1,1} + \|v_0\|_{H^1_{a_1}} + \|v_0\|_{H^1_{a_2}} = \varepsilon_0,\\
|c_{2,0} - c_{1,0}| + e^{-\frac{d_0}{2}(x_{2,0}-x_{1,0})} = \varepsilon_1,\\
c_{2,0} - c_{1,0} \ge -\varepsilon_1/2.
\end{gathered}$$

Then there exist positive numbers C and $\bar\varepsilon_1$ depending on a_1 and I such that if $\varepsilon_1 \in (0, \bar\varepsilon_1)$ and $\varepsilon_0 \le C\varepsilon_1^2$, there exist $c_{2,\infty} > c_{1,\infty} > 0$, $x_{i,\infty} \in \mathbb{R}$ $(i = 1,2)$, satisfying $c_{2,\infty} - c_{1,\infty} = O(\varepsilon_1)$ and

$$\left\| e^{a(x-c_{1,\infty}t-x_{1,\infty})}\left(u(t,\cdot) - \sum_{i=1,2}\varphi_{c_{i,\infty}}(\cdot - c_{i,\infty}t - x_{i,\infty})\right)\right\|_{H^1(\mathbb{R})} = O(e^{-C_1\varepsilon_1^{1-\eta}t}), \tag{4}$$

$$\left\| u(t,\cdot) - \sum_{i=1,2}\varphi_{c_{i,\infty}}(\cdot - c_{i,\infty}t - x_{i,\infty})\right\|_{L^\infty(\mathbb{R})} = O(t^{-\frac{1}{3}}) \tag{5}$$

as $t \to \infty$, where C_1 and η are positive constants with $\eta = O(\varepsilon_0)$. Furthermore if $3 < p < 5$, there exist $v_\infty \in L^2(\mathbb{R})$ and a positive number $\delta(\varepsilon_0)$ satisfying $\lim_{\varepsilon_0\to 0}\delta(\varepsilon_0) = 0$ and

$$\left\| u(t,\cdot) - \sum_{i=1,2}\varphi_{c_{i,\infty}}(\cdot - c_{i,\infty}t - x_{i,\infty})\right\|_{H^1(\mathbb{R})} \le \delta(\varepsilon_0), \tag{6}$$

$$\left\| u(t,\cdot) - \sum_{i=1,2}\varphi_{c_{i,\infty}}(\cdot - c_{i,\infty}t - x_{i,\infty}) - e^{-t\partial_x^3}v_\infty\right\| = o(1) \tag{7}$$

as $t \to \infty$.

Remark 1. For generic $p \in (1,5)$ including $p = 3$, the assumption (3) holds for for every $c > 0$ (see [15]).

To prove the result, we make use of the exponentially weighted space as in [15]. Since the linearized operator around the multi-pulse satisfies the spectral gap condition in the exponentially weighted space, we apply dynamical systems point of view as in [4]. However, we cannot directly apply the local-manifold theory to our problem. The difficult point of the problem is to show that dispersive part of the solution remains small in H^1. Since the orbital stability of multi-pulse solutions of (1) remains unknown if the equation is the non-integrable and the relative velocity between solitary waves are small, we use the scattering results to obtain H^1-estimate. This idea was first used by [2, 16] to show the asymptotic stability of solitary wave solutions to nonlinear Schrödinger equations. We show that the interaction of the dispersive part and the solitary waves becomes small as $t \to \infty$ and use the scattering result on (1) due to Hayashi-Naumkin [8, 9], which gave the large time asymptotics of small dispersive solutions of (1) with $p \geq 3$, to obtain H^1-estimate of the dispersive part of the solution.

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