On the global solvability for quasilinear hyperbolic equation of Kirchhoff type in exterior domain

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Let Ω be an exterior domain in $\mathbb{R}^n$ with smooth and compact boundary $\partial\Omega$. We will consider the global solvability of the following initial boundary value problem for the quasilinear hyperbolic equation of Kirchhoff type for initial data in the usual Sobolev spaces:

$$(1) \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = m(\|\nabla u\|_{L^2}^2)^2 \Delta u & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad \dfrac{\partial u}{\partial t}(x,0) = u_1(x) & x \in \Omega, \\ u(x,t) = 0 & x \in \partial\Omega, \quad t \geq 0. \end{cases}$$

Here

$$m(\lambda) \in C^1([0,\infty)) \quad \text{and} \quad \inf_{\lambda \geq 0} m(\lambda) (= m_0) > 0.$$

Since the coefficient $m(\|\nabla u\|^2)^2$ of Δ does not decay as $t \to \infty$, we cannot use the method used for proving the global solbavility of nonlinear wave equation $\dfrac{\partial^2 u}{\partial t^2} - \Delta u + F(x, \nabla u) = f(t,x)$.

The global solvability for analytic initial data was known in general setting. But that for initial data in the usual Sobolev space is known only for the following restricted cases. For a bounded domain Ω, Pokhozhaev [4] showed the global existence of solution for initial data in $\big(H^2(\Omega) \cap H_0^1(\Omega)\big) \times H_0^1(\Omega)$ for the special function $m(x) = (ax+b)^{-1}$ with positive a and b.

In case $\Omega = \mathbb{R}^n$, Greenberg-Hu [3] first showed the unique global solbavility for small initial data with some decay condition for $n = 1$. D'Ancona-Spagnolo [2] generalized their result for arbitrary n. For small initial data with some decay condition, they applied the stationary phase method to obtain the decay of the function $\dfrac{d}{dt} m(\|\nabla u\|^2)$ as $t \to \infty$, which yields the global solvability. In case $n = 1$, W. Rzymowski [8] improved Greenberg-Hu's result.

In case Ω is an exterior domain, Racke [7] showed the global solvability, and Heiming [6] improved Racke's result. They applied the method by Greenberg-Hu, D'Ancona-Spagnolo to the exterior problem by using the generalized Fourier transform instead of the Fourier transform, and obtained a condition on the generalized Fourier transform of the initial data sufficient for the unique global solvability.

The purpose of this talk is to give a sufficient condition on the initial data for the unique global solvability of the exterior problem for quasilinear hyperbolic equations of Kirchhoff type, by showing that the unique global solvability follows from a decay estimate of the solution of the wave equation in an exterior domain.

We use the following notations.

Let H be a Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let X be a Banach space with norm $\|\cdot\|_X$ which is dense in H. Let X' be the dual space of X with norm $\|\cdot\|_{X'}$ and $|(x, y)| \leq \|x\|_X \|y\|_{X'}$ for $x \in X$, $y \in H$. Let Z be a Banach space in X with norm $\|\cdot\|_Z$. Let A be a non-negative self-adjoint operator in H.

Let $Y_d = \{f \in L^1(\mathbb{R}); \|f\|_{Y_d} = \sup_{t\in\mathbb{R}}(1 + |t|)^d |f(t)| < \infty\}$ for $d \geq 0$.

Then we have the following result.

Theorem 1. *Let $\alpha \geq 1$. Let $Y = Y_d$ $(d > 1)$ or $Y = L^1(\mathbb{R})$. Assume that there exists a positive constant C such that for every $f \in D(A^\alpha) \cap Z$, the linear abstract hyperbolic equation*

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} + Av = 0 & t \in \mathbb{R}, \\ v(0) = f, \ \dfrac{\partial v}{\partial t}(0) = iA^{1/2} f \end{cases} \tag{2}$$

has a unique solution

$$v \in \bigcap_{i=0,1,2} C^i\left(\mathbb{R}; D(A^{1-i/2})\right)$$

and v satisfies the estimate

$$\left\| \left\| A^{1/2} v(t) \right\|_{X'} \right\|_Y \leq C \|f\|_Z. \tag{3}$$

Then there exists a positive δ such that the following holds: Let $\beta \geq \alpha + 1/2$. If $(u_0, u_1) \in D(A^\beta) \times D(A^{\beta - 1/2})$ satisfy $A^{1/2} u_0, u_1 \in Z$ and

$$\left(\left\| A^{1/2} u_0 \right\|_Z + \|u_1\|_Z \right) \left(\left\| A^{1/2} u_0 \right\|_X + \|u_1\|_X \right) \leq \delta,$$

then the quasilinear hyperbolic equation of Kirchhoff type

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + m\left(\|A^{1/2} u\|^2 \right)^2 Au = 0 & t > 0, \\ u(0) = u_0, \ \dfrac{\partial u}{\partial t}(0) = u_1 \end{cases} \tag{4}$$

has a unique global solution

$$u \in \bigcap_{i=0,1,2} C^i \left([0,\infty); D(A^{\beta-i/2})\right).$$

Furthermore, it holds that

$$\left\| \frac{d}{dt} m\left(\|A^{1/2}u(t)\|^2\right)\right\|_Y < \infty. \tag{5}$$

We will apply Theorem 1 to the mixed problem (1) in an exterior domain.

We use the following notations and definitions. Let $1 \le p \le \infty$, $s \ge 0$. Let

$$W_p^s(\Omega) = \left\{ f \;\middle|\; \exists g \in W_p^s(\mathbb{R}^n) \text{ such that } g|_\Omega = f \right\},$$

with the norm $\|f\|_{W_p^s(\Omega)} = \inf\left\{ \|g\|_{W_p^s(\mathbb{R}^n)} \;\middle|\; g|_\Omega = f\right\}$. If $p=2$, $W_2^s(\Omega)$ is denoted by $H^s(\Omega)$. Let $H_0^s(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. Then, if $s \in \mathbb{N}$, we have

$$H_0^s(\Omega) = \left\{ f \in H^s(\Omega) \;\middle|\; \partial_\nu^j f|_{\partial\Omega} = 0 \text{ for } j = 0,1,\cdots,s-1 \right\},$$

where ∂_ν denotes the outer normal derivative.

Let $L \ge 1$ be an integer. The initial data $(u_0,u_1) \in H^L(\Omega)\times H^{L-1}(\Omega)$ is said to satisfy the compatibility condition of order $L-1$ for $\Box u = 0$ in Ω, if $u_0 \in H^L(\Omega)$ and $u_1 \in H^{L-1}(\Omega) \cap H_0^{L-2}(\Omega)$ in case L is odd, and $u_0 \in H^L(\Omega) \cap H_0^{L-1}(\Omega)$ and $u_1 \in H_0^{L-1}(\Omega)$ in case L is even.

The exterior domain Ω is said to be "non-trapping" if the following is satisfied: Let $G(t,x,y)$ be the unique solution of the mixed problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u & & x \in \Omega, \quad t > 0, \\ u(x,0) = 0, & \dfrac{\partial u}{\partial t}(x,0) = \delta(x-y) & x \in \Omega, \\ u(x,t) = 0 & & x \in \partial\Omega, \quad t \ge 0, \end{cases}$$

where δ is the Dirac delta function and y is an arbitrary point in Ω. Let a and b be an arbitrary positive constants with $r_0 \le a \le b$. Then there exists a $T_0 > 0$ depending only on n, a, b and Ω such that

$$\int_\Omega G(t,x,y)v(y)dy \in C^\infty\left([T_0,\infty)\times\overline{\Omega_b}\right)$$

for every $v \in L_a^2(\Omega)$.

It is well known that, if the complement of Ω is convex, then Ω is non-trapping.

By using the cut-off argument by Shibata-Tsutsumi [9], which proves the L^p-L^q estimate for the solution of the linear wave equation in an exterior domain, we obtain the following:

Theorem 2. *Assume that* $n \geq 4$ *and* Ω *is non-trapping. Let* $2(n-1)/(n-3) < p \leq \infty$*, and let* q *be the real number such that* $1/p + 1/q = 1$*. Let* M *be an integer such that* $M \geq 1$*. Let* N *be an integer such that* $N \geq (n+1)(1/2 - 1/p)$*. Assume that initial data*

$$(\phi, \psi) \in \left(W_q^{(n+1)(1-2/p)+M}(\Omega) \cap H^{N+M}(\Omega) \right) \times \left(W_q^{(n+1)(1-2/p)+M-1}(\Omega) \cap H^{N+M-1}(\Omega) \right)$$

satisfy the compatibility condition of order $M+N-1$ *for* $\Box u = 0$ *in* Ω*. Then there exists a constant* $C = C(M, n, \Omega)$ *such that the unique global solution*

$$u \in \bigcap_{i=0,1,2} C^i \left([0, \infty); H^{M+N-i}(\Omega) \right)$$

of the mixed problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u & & x \in \Omega, \quad t > 0, \\ u(x,0) = \phi(x), & \dfrac{\partial u}{\partial t}(x,0) = \psi(x) & x \in \Omega, \\ u(x,t) = 0 & & x \in \partial\Omega, \quad t \geq 0, \end{cases}$$

satisfies the L^p*-*L^q *estimate*

$$\|u(t)\|_{W_p^M(\Omega)} \leq C(1+t)^{-(n-1)(1/2-1/p)} \left(\|\phi\|_{W_q^{(n+1)(1-2/p)+M}(\Omega)} + \|\psi\|_{W_q^{(n+1)(1-2/p)+M-1}(\Omega)} \right).$$

Remark. *Exactly in the same way as in the proof of Shibata-Tsutsumi [9], one can show that*

$$\|Du(t)\|_{W_p^{M-1}(\Omega)} \leq C(1+t)^{-(n-1)(1/2-1/p)} \left(\|\phi\|_{W_q^{2n+2+M}(\Omega)} + \|\psi\|_{W_q^{2n+1+M}(\Omega)} \right).$$

From Theorems 1 and 2 we obtain the following theorem:

Theorem 3. *Assume that* $n \geq 4$ *and* Ω *is non-trapping. Let* p *be a real number such that* $p > 2(n-1)/(n-3)$*, and* q *be the real number such that* $1/p + 1/q = 1$*. Let* L *be an integer such that* $L \geq (n+1)(1/2 - 1/p) + 2$*. Then there exists a positive constant* δ *such that the following holds: If the initial data*

$$(u_0, u_1) \in \left(H_0^L(\Omega) \cap W_q^{(n+1)(1-2/p)+2}(\Omega) \right) \times \left(H_0^{L-1}(\Omega) \cap W_q^{(n+1)(1-2/p)+1}(\Omega) \right)$$

satisfy

$$\left(\|u_0\|_{W_q^{(n+1)(1-2/p)+2}(\Omega)} + \|u_1\|_{W_q^{(n+1)(1-2/p)+1}(\Omega)}\right)\left(\|u_0\|_{W_q^1(\Omega)} + \|u_1\|_{L_q(\Omega)}\right) \le \delta,$$

then the mixed problem for quasilinear wave equation of Kirchhoff type has a unique global solution

$$u \in \bigcap_{i=0,1,2} C^i\left([0,\infty); H_0^{L-i}(\Omega)\right).$$

Furthermore, we have

$$\sup_{t\ge 0}(t+1)^{(n-1)(1/2-1/p)}\left|\frac{d}{dt}m\left(\|\nabla u(t)\|_{L^2}^2\right)\right| < \infty.$$

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