The initial value problem for Schrödinger equations on the torus

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In this note, we are concerned with the well-posedness of the initial value problems for linear Schrödinger-type equations of the form

$$Lu \equiv \partial_t u - i\Delta u + \vec{b}(x)\cdot\nabla u + c(x)u = f(t,x) \quad \text{in} \quad \mathbb{R}\times\mathbb{T}^n, \tag{1}$$
$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \tag{2}$$

and for semilinear Schrödinger equations of the form

$$\partial_t u - i\Delta u = F(u,\nabla u,\bar{u},\nabla\bar{u}) \quad \text{in} \quad \mathbb{R}\times\mathbb{T}^n, \tag{3}$$
$$u(0,x) = u_0(x) \quad \text{in} \quad \mathbb{T}^n, \tag{4}$$

where $u(t,x)$ is a complex valued unknown function of $(t,x) = (t,x_1,\dots,x_n) \in \mathbb{R}\times\mathbb{T}^n$, $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, $i = \sqrt{-1}$, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,\dots,n)$, $\nabla = (\partial_1,\cdots,\partial_n)$, $\Delta = \nabla\cdot\nabla$, and $\vec{b}(x) = (b_1(x),\dots,b_n(x))$, $c(x)$, $f(t,x)$ and $u_0(x)$ are given functions. Suppose that $b_1(x),\dots,b_n(x)$ and $c(x)$ are smooth functions on $\mathbb{T}^n$, and that $F(u,v,\bar{u},\bar{v})$ is a smooth function on $\mathbb{R}^{2+2n}$, and

$$F(u,v,\bar{u},\bar{v}) = O(|u|^2+|v|^2) \quad \text{near} \quad (u,v) = 0.$$

For the Euclidean case $x \in \mathbb{R}^n$, Mizohata proved in [8] that if the initial value problem (1)-(2) is L^2-well-posed, then

$$\sup_{(t,x,\omega)\in\mathbb{R}^{1+n}\times S^{n-1}} \left| \int_0^t \operatorname{Im}\vec{b}(x-\omega s)\cdot\omega ds \right| < +\infty, \tag{5}$$

where S^{n-1} is a unit sphere in $\mathbb{R}^n$ and $\vec{b}\cdot\xi = b_1\xi_1+\cdots+b_n\xi_n$. Moreover, he gave sufficient condition for L^2-well-posedness which is slightly stronger than (5). In particular, (5) is also sufficient condition for L^2-well-posedness when $n = 1$. Roughly speaking, (5) gives an upper bound of the strength of the real vector field $(\operatorname{Im}\vec{b}(x))\cdot\nabla$. In other words, if $(\operatorname{Im}\vec{b}(x))\cdot\nabla$ can be dominated by so-called local smoothing effect of $e^{it\Delta}$, then (5) must holds. After his results, many authors discovered similar sufficient conditions. Unfortunately, however, the characterization of L^2-well-posedness for (1)-(2) remains open except for one-dimensional case.

On the other hand, the periodic case is completely different from the Euclidean case. The local smoothing effect of $e^{it\Delta}$ fails because the flow of the hamiltonian vector field $2\xi\cdot\nabla$ is completely trapped. The purpose of this note is to present the necessary and sufficient condition of L^2-well-posedness of (1)-(2), and apply this condition to (3)-(4). We here give the definition of L^2-well-posedness.

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Definition 1. The initial-boundary value problem (1)-(2) is said to be L^2-well-posed if for any $u_0 \in L^2(\mathbb{T}^n)$ and $f \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{T}^n))$, (1)-(2) has a unique solution $u \in C(\mathbb{R}; L^2(\mathbb{T}^n))$.

Firstly, we present L^2-well-posedness results for linear equations.

Theorem 2. *The following conditions are mutually equivalent:*

1. (1)-(2) *is* L^2*-well-posed.*
2. *For* $x \in \mathbb{T}^n$ *and* $\alpha \in \mathbb{Z}^n$
$$\int_0^{2\pi} \operatorname{Im} \vec{b}(x - \alpha s) \cdot \alpha ds = 0. \tag{6}$$
3. *There exists a scalar function* $\phi(x) \in C^\infty(\mathbb{T}^n)$ *such that* $\nabla \phi(x) = \operatorname{Im} \vec{b}(x)$.

When $n = 1$, set $b(x) = b_1(x)$. The condition (6) is reduced to
$$\int_0^{2\pi} \operatorname{Im} b(x) dx = 0. \tag{7}$$
The condition (6) is the natural torus version of (5). More precisely, (6) is a special case of Ichinose's necessary condition of L^2-well-posedness discovered in [5]. On the other hand, the condition 3 corresponds to Ichinose's sufficient condition of L^2-well-posedness discovered in [6]. Theorem 2 makes us expect analogous results for nonlinear equations. In fact, we have local existence and local ill-posedness results as follows.

Theorem 3. *Let* $s > n/2 + 2$*. Suppose that there exists a smooth real-valued function* $\Phi(u, \bar{u})$ *on* $\mathbb{R}^2$ *such that for any* $u \in C^1(\mathbb{T}^n)$
$$\nabla \Phi(u, \bar{u}) = \operatorname{Im} \nabla_v F(u, \nabla u, \bar{u}, \nabla \bar{u}). \tag{8}$$
Then for any $u_0 \in H^s(\mathbb{T}^n)$*, there exists* $T > 0$ *depending on* $\|u_0\|_s$ *such that* (3)-(4) *possesses a unique solution* $u \in C([-T, T]; H^s(\mathbb{T}^n))$*. Furthermore, Let* $\{u_{0,k}\}$ *be a sequence of initial data belonging to* $H^s(\mathbb{T}^n)$*, and let* $\{u_k\}$ *be a sequence of corresponding solutions. If*
$$u_{0,k} \longrightarrow u_0 \quad in \quad H^s(\mathbb{T}^n) \quad as \quad k \to \infty,$$
then for any $m < s$
$$u_k \longrightarrow u \quad in \quad C([0, T]; H^m(\mathbb{T}^n)) \quad as \quad k \to \infty. \tag{9}$$

Theorem 4. *Suppose that there exists a holomorphic* n*-vector function*
$$\vec{G}(u) = (G_1(u), \cdots, G_n(u)), \quad u \in \mathbb{C}$$
such that $G(u) \not\equiv 0$*, and*
$$F(u, \nabla u, \bar{u}, \nabla \bar{u}) = \nabla \cdot \vec{G}(u) \tag{10}$$
for any $u \in C^1(\mathbb{T}^n)$*. Then* (3)-(4) *is* not *locally well-posed in the sense of Theorem* 3*.*

It seems to be hard to show the continuous dependence of the solution on the initial data because the gain of derivative of $e^{it\Delta}$ fails when $x \in \mathbb{T}^n$. To prove Theorem 4, we construct a sequence of solutions which are real-analytic in x by using the idea of the abstract Cauchy-Kowalewski theorem. Hence it is essential that $G(u)$ is holomorphic.

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