Damped wave equation in the subcritical case

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We study the one dimensional nonlinear damped wave equation

$$\begin{cases} v_{tt} + v_t - v_{xx} + v^{1+\sigma} = 0,\ x \in \mathbf{R},\ t > 0, \\ v(0,x) = \varepsilon v_0(x),\ v_t(0,x) = \varepsilon v_1(x), \end{cases} \tag{1}$$

in the sub critical case $\sigma \in (0,2)$, where $\varepsilon > 0$. Recently much attention was drawn to nonlinear wave equations with dissipative terms. For general dimensional case the problem is written

$$\begin{cases} v_{tt} + v_t - \Delta v + v^{1+\sigma} = 0,\ x \in \mathbf{R}^n,\ t > 0, \\ v(0,x) = \varepsilon v_0(x),\ v_t(0,x) = \varepsilon v_1(x). \end{cases} \tag{2}$$

In paper [24], it was proved global existence and large time decay estimates of solutions to the Cauchy problem for the damped wave equation with nonlinearities $\pm |v|^{1+\sigma}$ or $\pm |v|^{\sigma} v$, for the super critical case $\sigma > \frac{2}{n}$, if the initial data are sufficiently small and have a compact support. If we restrict our attention to low space dimensions we do not need to assume that the initial data have compact support. Indeed in the super critical case $\sigma > \frac{2}{n}$, the global existence in time of small solutions can be obtained by the method of paper [19] for $n = 1$. When $n = 3$ and $\partial^\alpha u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, $|\alpha| \le 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, problem (3) was considered in [21], [22] by making use of the fundamental solution of the linear problem and global existence of small solutions and large time decay estimates $\|u\|_{\mathbf{L}^q} \le C t^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$, $1 \le q \le \infty$ was obtained for $n = 3$. Later these requirements on the initial data were relaxed in [23] as follows $u_0 \in \mathbf{L}^1$, $\partial^\alpha u_0 \in \mathbf{L}^2$, $|\alpha| \le 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^2$, under the additional assumptions on σ and q such that $\sigma \le 5$, $q \le 6$ for the space dimension $n = 3$ and $q < \infty$ for the two dimensional case $n = 2$. For the case of higher dimensions $n = 4, 5$, global existence and $\mathbf{L}^q$ - time decay estimates for $\sigma \le q \le \frac{\sigma}{\sigma - 1}$ were obtained via Fourier analysis in paper [20], when the power of the nonlinearity σ is such that $1 + \frac{2}{n} < \sigma \le \frac{n+2}{n-2}$ and the initial data are small enough and satisfy $u_0, \partial^\alpha u_0 \in \mathbf{L}^1 \cap \mathbf{L}^{\frac{\sigma}{\sigma-1}}$, $\partial^\beta u_0 \in \mathbf{L}^2$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^{\frac{\sigma}{\sigma-1}}$, $\partial^\alpha u_1 \in \mathbf{L}^2$, $|\alpha| \le 1$, $|\beta| \le 2$. The blow-up results were proved in [24] for the case of nonlinearity $-|v|^{1+\sigma}$, with $\sigma < \frac{2}{n}$, when the initial data are such that $\int_{\mathbf{R}^n} v_0(x)\,dx > 0$, $\int_{\mathbf{R}^n} v_1(x)\,dx > 0$. Blow-up results for the critical and sub critical cases $\sigma \le \frac{2}{n}$ were obtained in [17]. Via the energy type estimates obtained in papers [19] and [15] it was proved in [13] that solutions of the nonlinear damped wave equation (1) in the case $\sigma > 1 + \frac{4}{n}, n = 1, 2, 3$ with arbitrary initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^1$, $u_1 \in \mathbf{L}^2 \cap \mathbf{L}^1$ (i.e. without smallness assumption on the initial data) have the same large time asymptotics as that for the linear heat equation $\partial_t - \partial_x^2$, that is

$$\|u(t) - MG_0(t)\|_{\mathbf{L}^p} = o\left(t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)}\right)$$

as $t \to \infty$,where $2 \le p \le \infty$ for $n = 1$, $2 \le p < \infty$ for $n = 2$, $2 \le p < 6$ for $n = 3$, and $G_0(t)$ is the heat kernel and M is a constant. If we do not assume the smallness condition on the data, as far as we know there is no result of asymptotic behavior of solutions to the problem for $\sigma \le 1 + \frac{4}{n}$ even if the order of nonlinearity is the super critical case $\sigma > 1 + \frac{n}{4}$. Recently the critical case $\sigma = \frac{2}{n}, n = 1, 2, 3$ was considered in paper [11] under some assumptitons on the data, where it was proved that small solutions of (2) have an additional time decay

$$\|v(t)\|_{\mathbf{L}^\infty} \le C \langle t \rangle^{-\frac{n}{2}} (1 + \log \langle t \rangle)^{-\frac{n}{2}},$$

where $\langle t \rangle = \sqrt{1 + t^2}$. In the case of sub-critical case we need to study the sharp asymptotics of solutions to linear problem in the weighted Sobolev space to obtain the desired result. For $n = 1$, $\mathbf{L}^p - \mathbf{L}^q$ asymptotics of the fundamental solutions was studied in detail, where $1 \le q \le p \le \infty$ in [18]. Note that similar behavior first was discovered for the nonlinear heat equation $v_t - \Delta v - v^{1+\sigma} = 0$ in the critical case $\sigma = \frac{2}{n}$, comparing with the linear heat equation, (see, e.g., [5]). For blow-up results we refer [4], [6], [16]. Large time behavior of solutions to the nonlinear heat equations in the sub critical cases $\sigma \in \left(0, \frac{2}{n}\right)$ was obtained in papers [2], [3], [7], [14], [25].

Taking $v = u_1$ and $(1 + \partial_x)^{-1} v_t = u_2$ we rewrite equation (1) in the form of a system of nonlinear evolutionary equations

$$u_t + \mathcal{N}(u) + \mathcal{L}u = 0 \tag{3}$$

for the vector $u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}$, with the initial data

$$u(0, x) = \widetilde{u}(x) \equiv \begin{pmatrix} \varepsilon v_0(x) \\ \varepsilon (1 + \partial_x)^{-1} v_1(x) \end{pmatrix},$$

where the linear part of system (3) is a pseudodifferential operator defined by the Fourier transformation as follows

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi \to x} L(\xi) \mathcal{F}_{x \to \xi} u,$$

with a matrix - symbol

$$L(\xi) = \{L_{jk}(\xi)\}|_{j,k=1,2} = \begin{pmatrix} 0 & -(1 + i\xi) \\ \frac{\xi^2}{1+i\xi} & 1 \end{pmatrix}$$

and the nonlinearity is defined dy

$$\mathcal{N}(u) = (1 + \partial_x)^{-1} \begin{pmatrix} 0 \\ u_1^{1+\sigma} \end{pmatrix},$$

with

$$\left(1+\partial_x\right)^{-1}=\overline{\mathcal{F}}_{\xi\to x}\left(1+i\xi\right)^{-1}\mathcal{F}_{x\to\xi}=e^{-x}\int_{-\infty}^{x}dx'e^{x'}.$$

We define the direct Fourier transformation $\mathcal{F}_{x\to\xi}$ by

$$\hat{u}\left(\xi\right)\equiv\mathcal{F}_{x\to\xi}u=\left(2\pi\right)^{-\frac{1}{2}}\int_{\mathbf{R}}e^{-i\xi x}u\left(x\right)dx,$$

then the inverse Fourier transformation $\overline{\mathcal{F}}_{\xi\to x}$ is

$$\check{u}\left(x\right)\equiv\overline{\mathcal{F}}_{\xi\to x}u=\left(2\pi\right)^{-\frac{1}{2}}\int_{\mathbf{R}}e^{ix\xi}u\left(\xi\right)d\xi.$$

Denote by

$$\lambda_1\left(\xi\right)=\frac{1}{2}+\frac{1}{2}\sqrt{1-4\xi^2},\lambda_2\left(\xi\right)=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi^2}$$

the eigenvalues of the matrix $L\left(\xi\right)$. Note that the matrix

$$Q\left(\xi\right)=\left(\begin{array}{cc}Q_{11}\left(\xi\right) & Q_{12}\left(\xi\right)\\ Q_{21}\left(\xi\right) & Q_{22}\left(\xi\right)\end{array}\right)=\left(\begin{array}{cc}1+i\xi & 1+i\xi\\ -\lambda_1\left(\xi\right) & -\lambda_2\left(\xi\right)\end{array}\right)$$

and

$$Q^{-1}\left(\xi\right)=\frac{1}{\left(1+i\xi\right)\left(\lambda_1\left(\xi\right)-\lambda_2\left(\xi\right)\right)}\left(\begin{array}{cc}-\lambda_2\left(\xi\right) & -\left(1+i\xi\right)\\ \lambda_1\left(\xi\right) & 1+i\xi\end{array}\right)$$

diagonalize the matrix $L\left(\xi\right)$, i.e.

$$Q^{-1}\left(\xi\right)L\left(\xi\right)Q\left(\xi\right)=\left(\begin{array}{cc}\lambda_1\left(\xi\right) & 0\\ 0 & \lambda_2\left(\xi\right)\end{array}\right).$$

Consider the system of ordinary differential equations with constant coefficients depending on the parameter $\xi\in\mathbf{R}$

$$\frac{d}{dt}\widehat{u}\left(t,\xi\right)+L\left(\xi\right)\widehat{u}\left(t,\xi\right)=0. \tag{4}$$

Multiplying system (4) by $Q^{-1}\left(\xi\right)$ from the left and changing $\widehat{u}\left(t,\xi\right)=Q\left(\xi\right)w\left(t,\xi\right)$ we diagonalize system (4)

$$\frac{d}{dt}\left(\begin{array}{c}w_1\left(t,\xi\right)\\ w_2\left(t,\xi\right)\end{array}\right)=-\left(\begin{array}{cc}\lambda_1\left(\xi\right) & 0\\ 0 & \lambda_2\left(\xi\right)\end{array}\right)\left(\begin{array}{c}w_1\left(t,\xi\right)\\ w_2\left(t,\xi\right)\end{array}\right),$$

whence integrating with respect to time $t\geq0$ we find

$$\left(\begin{array}{c}w_1\left(t,\xi\right)\\ w_2\left(t,\xi\right)\end{array}\right)=\left(\begin{array}{cc}e^{-t\lambda_1\left(\xi\right)} & 0\\ 0 & e^{-t\lambda_2\left(\xi\right)}\end{array}\right)\left(\begin{array}{c}w_1\left(0,\xi\right)\\ w_2\left(0,\xi\right)\end{array}\right)$$

Returning to the solution $\widehat{u}\left(t,\xi\right)$ we get

$$\begin{aligned}\widehat{u}\left(t,\xi\right) &= \left(\begin{array}{c}\widehat{u}_1\left(t,x\right)\\ \widehat{u}_2\left(t,x\right)\end{array}\right)=Q\left(\xi\right)\left(\begin{array}{c}w_1\left(t,\xi\right)\\ w_2\left(t,\xi\right)\end{array}\right)\\ &= Q\left(\xi\right)\left(\begin{array}{cc}e^{-t\lambda_1\left(\xi\right)} & 0\\ 0 & e^{-t\lambda_2\left(\xi\right)}\end{array}\right)Q^{-1}\left(\xi\right)\left(\begin{array}{c}\widehat{u}_0\left(\xi\right)\\ \widehat{u}_1\left(\xi\right)\end{array}\right)\\ &= e^{-tL\left(\xi\right)}\left(\begin{array}{c}\widehat{u}_0\left(\xi\right)\\ \widehat{u}_1\left(\xi\right)\end{array}\right),\end{aligned}$$

where the fundamental Cauchy matrix has the form

$$\begin{aligned} e^{-tL(\xi)} &= Q(\xi) \begin{pmatrix} e^{-t\lambda_1(\xi)} & 0 \\ 0 & e^{-t\lambda_2(\xi)} \end{pmatrix} Q^{-1}(\xi) \\ &= \frac{1}{\sqrt{1-4\xi^2}} \begin{pmatrix} -\lambda_2(\xi) & -(1+i\xi) \\ \frac{\xi^2}{1+i\xi} & \lambda_1(\xi) \end{pmatrix} e^{-t\lambda_1(\xi)} \\ &\quad + \frac{1}{\sqrt{1-4\xi^2}} \begin{pmatrix} \lambda_1(\xi) & 1+i\xi \\ -\frac{\xi^2}{1+i\xi} & -\lambda_2(\xi) \end{pmatrix} e^{-t\lambda_2(\xi)}. \end{aligned}$$

We rewrite the Cauchy problem (3) in the form of the integral equation

$$u(t) = \mathcal{G}(t)\,\widetilde{u} - \int_0^t \mathcal{G}(t-\tau)\,\mathcal{N}(u)(\tau)\,d\tau, \tag{5}$$

where the Green operator $\mathcal{G}(t)\psi = \overline{\mathcal{F}}_{\xi\to x}\left(e^{-tL(\xi)}\hat{\psi}(\xi)\right)$. So by the solution of the Cauchy problem (3) we always understand the solution $u(t,x)$ of the corresponding integral equation (5), belonging to $\mathbf{C}^0([0,\infty);\mathbf{X})$ with an appropriate choice of a functional space $\mathbf{X}$.

In the present paper we prove the following result. Denote

$$\mathbf{L}^{1,a} = \left\{\phi \in \mathbf{L}^1 : \|\phi\|_{\mathbf{L}^{1,a}} = \|\langle\cdot\rangle^a \phi\|_{\mathbf{L}^1} < \infty\right\}.$$

Theorem 1. *We assume that the initial data* $\widetilde{u} \in \left(\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right)^2$, $a \in (0,1)$, *and the mean value*

$$\theta = \varepsilon \int_{\mathbf{R}} \left(\widetilde{u}_1(x) + \widetilde{u}_2(x)\right) dx > 0.$$

Then there exists a positive ε *such that the Cauchy problem for equation (3) has a unique mild solution* $u(t,x) \in \left(\mathbf{C}\left([0,\infty);\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right)\right)^2$ *satisfying the following time decay estimate*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon \langle t\rangle^{-\frac{1}{\sigma}}$$

for large $t > 0$ *and any* $\sigma \in \left(2-\varepsilon^3, 2\right)$. *Furthermore the asymptotic formula*

$$u(t,x) = e_1\left((t\eta)^{-\frac{1}{\sigma}} V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right)\right),$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* $\gamma = \frac{1}{2}\min\left(a, 1-\frac{\sigma}{2}\right)$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V \in \mathbf{L}^{1,a} \cap \mathbf{L}^\infty$ *is the solution of the integral equation*

$$\begin{aligned} V(\xi) &= \frac{1}{(4\pi)^{\frac{1}{2}}} e^{-\frac{\xi^2}{4}} \\ &\quad - \frac{1}{\eta(4\pi)^{\frac{1}{2}}} \int_0^1 \frac{dz}{z(1-z)^{\frac{1}{2}}} \int_{\mathbf{R}} e^{-\frac{(\xi - y\sqrt{z})^2}{4(1-z)}} F(y)\,dy, \end{aligned} \tag{6}$$

where

$$\eta = \frac{\sigma}{1-\frac{\sigma}{2}} \int_{\mathbf{R}} V^{1+\sigma}(y)\,dy$$

and

$$F(y) = V^{1+\sigma}(y) - V(y)\int_{\mathbf{R}} V^{1+\sigma}(\xi)\, d\xi.$$

Remark 1. *As a consequence of Theorem 1 we have the following asymptotics for the damped wave equation (1)*

$$v(t,x) = (t\eta)^{-\frac{1}{\sigma}} V\left(\frac{x}{\sqrt{t}}\right) + O\left(t^{-\frac{1}{\sigma}-\gamma}\right),$$

for $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$ *if the initial data* $v_0, (1+\partial_x)^{-1} v_1 \in \left(\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right)^2$, $a \in (0,1)$, *are such that the mean value*

$$\int_{\mathbf{R}} (v_0(x) + v_1(x))\, dx > 0.$$

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