EXISTENCE AND REGULARITY FOR THE EVOLUTION OF HIGHER-DIMENSIONAL $H$–SYSTEMS

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Abstract Let $m \geq 2$ be a positive integer and $\Omega$ be a bounded domain in $m$–dimensional Euclidean space $\mathbb{R}^m$ with smooth boundary $\partial \Omega$. For a real number $H$, the surface of constant mean curvature $H$ in $\mathbb{R}^{m+1}$ is prescribed by the nonlinear degenerate elliptic systems of second order partial differential equations, called “$H$–system”,

$$-\text{div} \left( |\nabla u|^{m-2} \nabla u \right) = m \frac{m}{2} H \nabla_1 u \wedge \cdots \wedge \nabla_m u$$ (1.1)

for a map $u(x) = (u^1(x), \ldots, u^{m+1}(x))$ defined for $x = (x_1, \ldots, x_m) \in \Omega$ with values into $\mathbb{R}^{m+1}$, where $\nabla_\alpha = \frac{\partial}{\partial x^\alpha}$, $\alpha = 1, \ldots, m$, $\nabla u$ is the spatial gradient of a map $u$, $\nabla u = (\nabla_\alpha u^i)$, and the cross product $w_1 \wedge \cdots \wedge w_m : \mathbb{R}^{m+1} \oplus \cdots \oplus \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$ is defined by the property that $w \cdot w_1 \wedge \cdots \wedge w_m = \det W$ for all vectors $w, w_i \in \mathbb{R}^{m+1}$, $i = 1, \ldots, m$, and for the $(m+1) \times (m+1)$–matrix $W$ having the first row $(w^1, \ldots, w^{m+1})$ and the $i$–th row $(w^i_{i-1}, \ldots, w^i_{m+1})$, $i = 2, \ldots, m+1$. Here and in what follows, the notation and the summation notation over repeated indices is adopted.

We call a map $u : \Omega \rightarrow \mathbb{R}^{m+1}$ conformal if

$$\nabla_\alpha u \cdot \nabla_\beta u = \lambda^2 \delta_{\alpha \beta} \quad \alpha, \beta = 1, \ldots, m, \quad (1.2)$$

hold in $\Omega$ for some real-valued function $\lambda$ which does not vanish in $\mathbb{R}^m$. If a map $u$ is of $C^2$–class and conformal, then it is seen that $u$ actually defines a hypersurface $u(\Omega)$ in $\mathbb{R}^{m+1}$ which has constant mean curvature $H$ at every point $u(x) \in u(\Omega)$, $x \in \Omega$. Now we consider the Dirichlet boundary value problem for (1.1) with boundary value $u_0$ which is a given smooth map defined on $\overline{\Omega}$ with values in $\mathbb{R}^{m+1}$.

Note that the equation (1.1) has a variational structure. In fact, a solution of (1.1) gives a surface of least area enclosing a given volume. We call such surfaces as above soap–bubbles. We can recognize a solution of the Dirichlet problem for (1.1) to be a critical point of the variational problem, which is to minimize the variational functional, called “$m$–energy”,

$$I(u) = \int_{\Omega} \frac{1}{m} |\nabla u|^m \, dx, \quad (1.3)$$

under the constraint that the quantity, called “volume-functional”,

$$V(u) = \frac{1}{m+1} \int_{\Omega} u \cdot \nabla_1 u \wedge \cdots \wedge \nabla_m u \, dx \quad (1.4)$$

is prescribed by $V(u) = \text{a given constant}$, where $V(u)$ is interpreted as the algebraic volume enclosed between the surface $u(\Omega)$ and a fixed surface $u_0(\partial \Omega)$ spanning the curve $u_0(\partial \Omega)$ defined by the Dirichlet data $u_0$. Observe that (1.1) is the Euler–Lagrange equation of the functional

$$E(u) = I(u) - m \frac{m}{2} HV(u).$$

The one approach to look for a solution of the Dirichlet boundary value problem for (1.1) is to exploit the evolution for (1.1). Consider the Cauchy–Dirichlet problem: For a map $u : \Omega_\infty = (0, \infty) \times \Omega \rightarrow \mathbb{R}^{m+1}$, $u(z) = (u^1(z), \ldots, u^n(z))$, $z = (t, x) \in \Omega_\infty$,

$$\partial_t u - \text{div} \left( |\nabla u|^{m-2} \nabla u \right) = m \frac{m}{2} H \nabla_1 u \wedge \cdots \wedge \nabla_m u \quad \text{in} \ \Omega_\infty, \quad (1.5)$$

$$u = u_0 \quad \text{on} \ \{ t = 0 \} \times \overline{\Omega} \cup (0, \infty) \times \partial \Omega. \quad (1.6)$$
$E(u) = I(u) - m^2 HV(u)$. We report a global existence and a regularity of a weak solution of (1.5) and (1.6) for a smooth data having a “small” image. Our main result is the following.

**Theorem 1** Suppose that $u_0$ be a $W^{1,m}-$map defined on $\Omega$ with values in $\mathbb{R}^{m+1}$ satisfying the “smallness” condition $|H| \sup_{\Omega} |u_0| < 1$. Then, there exists a weak solution $u \in L^\infty(0, \infty; W^{1,m}(\Omega, \mathbb{R}^{m+1})) \cap W^{1,2}(0, \infty; L^2(\Omega, \mathbb{R}^{m+1}))$ of (1.5) and (1.6) such that $\sup_{\Omega} |u(t)| \leq \sup_{\Omega} |u_0|$ holds for any $t \geq 0$ and

$$\int_{(0,T) \times \Omega} |\partial_t u|^2 dz + \sup_{0 \leq t \leq T} E(u(t)) \leq E(u_0)$$

holds for all $T > 0$. The solution $u$ also satisfies the initial condition, $|u(t) - u_0|_{W^{1,m}(\Omega)} \to 0$ as $t \to 0$, and boundary condition $u(t) = u_0$ on $\partial \Omega$ in the trace sense in $W^{1,m}(\Omega, \mathbb{R}^{m+1})$ for almost every $t \in (0, \infty)$.

**Theorem 2** Suppose that $u_0$ be a $C^2-$map defined on $\overline{\Omega}$ with values in $\mathbb{R}^{m+1}$ satisfying the “smallness” condition $|H| \sup_{\Omega} |u_0| < \frac{1}{2}$. There exists a positive constant $\alpha < 1$ such that the weak solution obtained is locally Hölder continuous in $(0, \infty) \times \overline{\Omega}$ with an exponent $\alpha$ on the parabolic metric $|t|^\frac{1}{2} + |x|$ and the Hölder constant is bounded by a positive constant depending only on $m, \alpha, \partial \Omega, |H|$ and the $C^2-$norm of the data $u_0$. The gradient of the solution is also locally Hölder continuous in $(0, \infty) \times \Omega$ with an exponent $\alpha$ on the usual parabolic metric $|t|^\frac{1}{2} + |x|$ and the Hölder constant is bounded by a positive constant depending only on $m, \alpha, |H|$ and $I(u_0)$.

To prove the existence, we use a time-discrete approximation which consists of the minimization of a family of variational functionals, of which the Euler-Lagrange equations are the time-discrete elliptic partial differential equations of Rothe-type for (1.5). In the study of regularity of a weak solution obtained above, we apply the regularity theorem for the evolutionary $p-$Laplacian systems with critical growth on the gradient.

**References**


