An Inverse Problem for the Wave Equation in Plane-stratified Media

Sei Nagayasu

Department of Mathematics, Graduate School of Science, Osaka University
Machikaneyama 1-16, Toyonaka, Osaka 560-0043, Japan
E-mail: `sei@cr.math.sci.osaka-u.ac.jp`

Assume that there exist media which have singularities in a half-space. We investigate the singularities of the media by causing an artificial shock at a certain point near the boundary of the half-space and by observing the behavior of waves on the boundary. These problems for wave equations and elastic equations were studied by Rakesh [2] and Wang [4], for example. In these paper, they use the "linearization" method, that is, they assume the smallness of the singularities of the media and so on (Sacks-Symes [3]).

We discuss the case when the singularities may be large. We specialize the situation and discuss the following problem: Assume that two media, Medium 1 and Medium 2, are laying in the half-space, and the interface wall is parallel to the boundary of a half-space (see Figure 1). We assume that the speed of waves in Medium 1 and the way of the reflection by the boundary are known, but the width of Medium 1, the speed of waves in Medium 2, and interface and transmission conditions are unknown. In this situation, we try to identify these unknown things by using the known data or the data which can be observed near the boundary.

Now, we introduce the notation and formulate the problem above. Suppose $n \geq 2$. Let us write $x' = (x_1, \dots, x_{n-1})$, and $x'' = (x_2, \dots, x_n)$ for the coordinate $x = (x_1, \dots, x_n)$ in $\mathbb{R}^n$. The variable x_1 plays the role of the time and x'' the physical space.

Let $h > 0$ and $\Omega_1 := \{x'' \in \mathbb{R}^{n-1} : 0 < x_n < h\}$, $\Omega_2 := \{x'' \in \mathbb{R}^{n-1} : x_n > h\}$. We set $D_{x_j} := (1/i)(\partial/\partial x_j)$. Let a_k be positive real number and

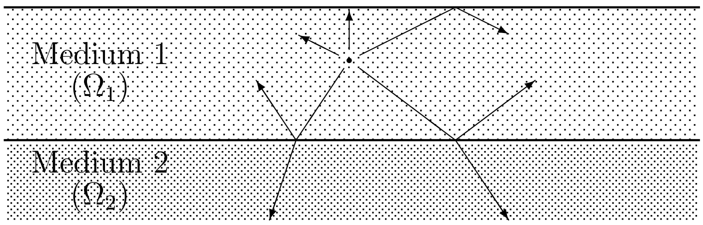

Figure 1:

set $P_k(D_x) := a_k^2(D_{x_2}^2 + \cdots + D_{x_n}^2) - D_{x_1}^2$ for $k = 1, 2$. The positive number a_k describes the speed of waves in Ω_k. Let $Q(D_x)$ be a partial differential operator with constant coefficients of first order, and write $Q(D_x) = q_1 D_{x_1} + \cdots + q_n D_{x_n} + q_0$. Furthermore we assume the coefficient q_n for D_{x_n} of $Q(D_x)$ is not zero. Let b_1, b_2, c_1, c_2 be constants. Suppose $0 < y_n < h$. Set $y'' := (0, \ldots, 0, y_n) \in \mathbb{R}^{n-1}$ and $y := (0, y'') \in \mathbb{R}^n$.

We discuss the following equations:

$$P_1(D_x)u(x) = \delta(x - y), \quad x_1 \in \mathbb{R},\ x'' \in \Omega_1, \tag{1}$$
$$P_2(D_x)u(x) = 0, \quad x_1 \in \mathbb{R},\ x'' \in \Omega_2, \tag{2}$$
$$Q(D_x)u(x)|_{x_n=0} = 0, \quad x' \in \mathbb{R}^{n-1}, \tag{3}$$
$$b_1 u(x)|_{x_n=h_-} = c_1 u(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}, \tag{4}$$
$$b_2 D_{x_n} u(x)|_{x_n=h_-} = c_2 D_{x_n} u(x)|_{x_n=h_+}, \quad x' \in \mathbb{R}^{n-1}. \tag{5}$$

These equations describe the situation that the initial data is the delta function at a point y'' in Ω_1 at time $x_1 = 0$ with the boundary condition (3) and the interface or transmission conditions (4) and (5). We assume that the mixed problem for the operator system $\{P_1(D_x), P_2(D_x); Q(D_x); b_1, c_1; b_2 D_{x_n}, c_2 D_{x_n}\}$ is $\mathcal{E}$ well-posed.

The following main result says that we can reconstruct the width h of Ω_1, the speed of waves a_2 in Ω_2 and the interface or transmission condition to a certain degree from the observation data of u near the boundary $\{x_n = 0\}$ when the speed a_1 on Ω_1 and the boundary condition are known.

Main Result. *Let a_1, $Q(D_x)$, y_n be given. Assume that observation data $u(x)|_{x_n=0}$ are given, where $u(x)$ denote the solution of equations* (1)–(5). *Then the constants h, a_2 and the ratio of $b_1 c_2$ to $b_2 c_1$ are reconstructed in the following sense:*

- *The constant h is reconstructed by using known data a_1, $Q(D_x)$, y_n and the observation data $u(x)|_{x_n=0}$ unless $u(x)|_{x_n=0} \equiv \widetilde{u}(x)|_{x_n=0}$. Here $\widetilde{u}$ is the wave in the situation that only one medium Medium 1 is laying in the half-space, that is, the solution of*

$$P_1(D_x)\widetilde{u}(x) = \delta(x-y), \quad x \in \mathbb{R}^n,$$
$$Q(D_x)\widetilde{u}(x)|_{x_n=0} = 0, \quad x' \in \mathbb{R}^{n-1}.$$

- *Suppose $n \geq 3$. Then the ratio of b_1c_2 to b_2c_1 is reconstructed by a_1, $Q(D_x)$, y_n and $u(x)|_{x_n=0}$. If $b_1b_2c_1c_2 \neq 0$ then a_2 is also reconstructed.*

- *Suppose $n = 2$. Then the ratio of b_1c_2 to $a_2b_2c_1$ is reconstructed by a_1, $Q(D_x)$, y_n and $u(x)|_{x_n=0}$.*

We prove this main result by using the solution formula of the problem (1)–(5), given by Matsumura [1] as the fundamental solution of the mixed problem for the operator system $\{P_1(D_x),\ P_2(D_x);\ Q(D_x);\ b_1,\ c_1;\ b_2D_{x_n},\ c_2D_{x_n}\}$.

References

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