Large time behavior of solutions to Schrödinger equations with nonlinearity of integral type

T. Ozawa
Department of Mathematics, Hokkaido University
Sapporo 060-0810

This talk is based on my recent joint work with Y. Yamazaki [32]. We study the Cauchy problem for the nonlinear Schrödinger equation with interaction described by the integral with respect to one direction of the intensity in two space dimensions:

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \tag{NLS}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \ni (t, (x, y)) \mapsto u(t, x, y) \in \mathbb{C}$, Δ is the Laplacian in space $\mathbb{R}^2$, and $f(u)$ is the nonlinear interaction given by

$$(f(u))(t, x, y) = \lambda u(t, x, y) \int_{-\infty}^{x} |u(t, x', y)|^2 \mathrm{d}x'$$

with $\lambda \in \mathbb{R}$. For $\alpha, \beta \geq 0$, we define $\dot{H}^{\alpha,\beta} = L_y^2 \dot{H}_x^\alpha \cap L_x^2 \dot{H}_y^\beta$ with norm $||u; \dot{H}^{\alpha,\beta}|| = \mathrm{Max}(||u; L_y^2 \dot{H}_x^\alpha||, ||u; L_x^2 \dot{H}_y^\beta||)$.

Theorem 1. *Let α and β satisfy $0 \leq \alpha < 1/2$ and $\beta \geq 0$. Let $\phi \in \mathcal{F}(\dot{H}^{\alpha,\beta})$ where $\mathcal{F}$ is the Fourier transform. The (NLS) has a unique solution*

$$u \in X_{\mathrm{loc}} \equiv C(\mathbb{R}; L^2) \cap \bigcap_{0 \leq 2/q = 1/2 - 1/r \leq 1/2} L_{\mathrm{loc}}^q(\mathbb{R}; L_x^2 L_y^r).$$

Moreover, $|J_x|^\alpha u$, $|J_y|^\beta u \in X_{\mathrm{loc}}$, where

$$\begin{aligned}
&|J_x|^\alpha = U_x(t)|x|^\alpha U_x(-t) = M_x(t)(-t^2\Delta_x)^{\alpha/2} M_x(-t),\\
&|J_y|^\beta = U_y(t)|y|^\beta U_y(-t) = M_y(t)(-t^2\Delta_y)^{\beta/2} M_y(-t),\\
&U_x(t) = \exp(\frac{it}{2}\Delta_x), U_y(t) = \exp(\frac{it}{2}\Delta_y),\\
&M_x(t) = \exp(\frac{ix^2}{2t})\cdot, M_y(t) = \exp(\frac{iy^2}{2t})\cdot .
\end{aligned}$$

Basic estimates for the proof of Theorem 1 are based on the fractional Leibniz rule [25], the generalized Hölder inequality in the Lorentz spaces [29,30], and the boundedness of the Hilbert transform and the Riesz potential in the Lorentz space [27,36].

To describe the large time behavior of solutions of (NLS) with small Cauchy data, we introduce modified free dynamics for $\phi_\pm \in L^2 \cap \mathcal{F}(L^2_x L^\infty_y)$:

$$\begin{aligned} v_1^\pm(t) &= U(t)\exp(-iS_\pm(t,-i\nabla))\phi_\pm, \\ v_2^\pm(t) &= U(t)M(-t)\exp(-iS_\pm(t,-i\nabla))\phi_\pm \\ &= M(t)D(t)\exp(-iS_\pm(t,\cdot))\hat{\phi}_\pm, \\ v_3^\pm(t) &= \exp(-iS_\pm(t,t^{-1}x,t^{-1}y))U(t)\phi_\pm \\ &= M(t)D(t)\exp(-iS_\pm(t,\cdot))\mathcal{F}M(t)\phi_\pm, \end{aligned}$$

where

$$\begin{aligned} &U(t) = \exp\left(\frac{it}{2}\Delta\right) = M(t)D(t)\mathcal{F}M(t), \\ &M(t) = \exp\left(\frac{i}{2t}(x^2+y^2)\right)\cdot, \\ &(D(t)\psi)(x,y) = (it)^{-1}\psi(t^{-1}x,t^{-1}y), \\ &S_\pm(t,x,y) = \pm\lambda\int_{-\infty}^{x}|\hat{\phi}_\pm(x',y)|^2 \mathrm{d}x' \log|t|. \end{aligned}$$

For $\rho, \rho_0 > 0$ we define

$$B(\rho,\rho_0) = \{\phi \in L^2 \cap \mathcal{F}(\dot{H}^{\alpha,\beta}); ||\phi : L^2|| \le \rho_0, ||\phi; \mathcal{F}(\dot{H}^{\alpha,\beta})|| \le \rho\}.$$

Theorem 2. *Let α and β satisfy $0 < \alpha < 1/2 < \beta < 1$. Then for any $\rho_0 > 0$ there exists $\rho > 0$ such that for any $\phi \in B(\rho,\rho_0)$ the solution u given by Theorem 1 satisfies*

$$\begin{aligned} &u \in X \equiv C(\mathbb{R};L^2) \cap \bigcap_{0\le 2/q = 1/2-1/r \le 1/2} L^q(\mathbb{R}; L^2_x L^r_y), \\ &|J_x|^\alpha u, |J_y|^\beta u \in X, \\ &||u(t); L^2_x L^\infty_y|| = O(|t|^{-1/2}) \quad as \quad t \to \pm\infty. \end{aligned}$$

Moreover, there exist unique $\phi_\pm \in L^2 \cap \mathcal{F}(L^2_x L^\infty_y)$ such that for $\varepsilon > 0$ sufficiently small

$$||\mathcal{F}U(-t)u(t) - \exp(-iS_\pm(t,\cdot))\hat{\phi}_\pm; L^2 \cap L^2_x L^\infty_y|| = O(|t|^{-\varepsilon})$$

as $t \to \pm\infty$. Furthermore,

$$||u(t) - v_j(t); L^2|| = O(|t|^{-\varepsilon})$$

as $t \to \pm\infty$, for $j = 1, 2$, and

$$||u(t) - v_3(t); L^2|| \to 0$$

as $t \to \pm\infty$.

We use the method of Hayashi and Naumkin [15,17]. The following ingredients are new and necessary to provide improvements, however. First, our method depends exclusively on a contraction argument and is independent of a contradiction argument in [15,17,18]. Secondly, our method depends exclusively on the generators of Galilei transforms and is independent of the usual regularity argument. This enables us not to impose any regularity assumption on the Cauchy data. Thirdly, our argument treats the L^2 norm and homogeneously weighted norms separately for the Cauchy data as well as for solutions. This enable us not to impose smallness of the L^2 norm of the Cauchy data. For instance, data of the form $\varepsilon^{-1}\psi(\varepsilon^{-1}x, \varepsilon^{-1}y)$ with $\varepsilon > 0$ sufficiently small and $\psi \in L^2 \cap \mathcal{F}(\dot{H}^{\alpha,\beta})$ fall within the scope of Theorem 2.

References

[1] J. B. Baillon, T. Cazenave and M. Figueira, Equation de Schrödinger avec non-linéarité intégrale, C. R. Acad. Sci. Paris **284** (1977), 939-942.

[2] D. J. Benney and G. L. Roskes, Wave instabilities, Stud. Appl. Math. **48** (1969), 377-385.

[3] J. Bergh and J. Löfström, "Interpolation spaces," Springer-Verlag, Berlin, Heidelberg, New York, 1976.

[4] T. Cazenave, Equations de Schrödinger non linéaires en dimension deux, Proc. Royal Soc. Edinburgh **84** (1979), 327-346.

[5] T. Cazenave, "An Introduction to Nonlinear Schrödinger Equations," Textos de Métodos Matemáticos **22**, Instituto de Mathemática, Rio de Janeiro, 1989.

[6] A. Davey and K. Stewartson, On three-dimensional packets of surface waves, Proc. Royal Soc. London **A 338** (1974), 101-110.

[7] V. D. Djordjevic and L. G. Redekopp, On two-dimensional packets of capillary-gravity waves, J. Fluid Mech. **79** (1977), 703-714.

[8] J.-M. Ghidaglia and J.-C. Saut, On the initial value problem for the Davey-Stewartson systems, Nonlinearity **3** (1990), 475-506.

[9] J. Ginibre, An introduction to nonlinear Schrödinger equations, in "Nonlinear Waves" (R. Agemi, Y. Giga and T. Ozawa, Eds.), GAKUTO International Series, Math. Sci. Appl. **10**, 1997.

[10] J. Ginibre and T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space in dimension $n \geq 2$, Commun. Math. Phys. **151** (1993), 619-645.

[11] J. Ginibre, T. Ozawa and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Phys. Théor. **60** (1994), 211-239.

[12] J. Ginibre and G. Velo, Long range scattering and modified wave operators for some Hartree type equations, I:Rev. Math. Phys. **12** (2000), 361-429; II: Ann. Inst. Henri Poincaré **1** (2000), 753-800.

[13] N. Hayashi, Local existence in time of small solutions to the Davey-Stewartson system, Ann. Inst. Henri Poincaré, Phys. Théor. **65** (1996), 313-366.

[14] N. Hayashi, Local existence in time of solutions to the elliptic-hyperbolic Davey-Stewartson system without smallness condition on the data, J. Analyse Math. **73** (1997), 133-164.

[15] N. Hayashi and P.I. Naumkin, Asymptotic for large time of solutions to the nonlinear Schrödinger and Hartree equations, Amer. J. Math. **120** (1998), 369-389.

[16] N. Hayashi and P.I. Naumkin, On the Davey-Stewartson and Ishimori systems, Math. Phys. Anal. Geom. **2** (1999), 53-81.

[17] N. Hayashi and P.I. Naumkin, Asymptotic behavior in time of solutions to the derivative nonlinear Schrödinger equation, Ann. Inst. Henri Poincaré, Phys. Théor. **68** (1998), 159-177.

[18] N. Hayashi and P.I. Naumkin and T. Ozawa, Scattering theory for the Hartree equation, SIAM J. Math. Anal. **29** (1998), 1256-1267.

[19] N. Hayashi and T. Ozawa, Scattering theory in the weighted L^2 $(\mathbb{R}^n)$ spaces for some schrödinger equations, Ann. Inst. Henri Poincaré, Phys. Théor. **48** (1988), 17-37.

[20] N. Hayashi and T. Ozawa, Modified wave operators for the derivative nonlinear Schrödinger equations, Math. Annalen **298** (1994), 557-576.

[21] N. Hayashi and T. Ozawa, Schrödinger equations with nonlinearity of integral type, Discrete and Continuous Dynamical Systems **1** (1995), 475-484.

[22] N. Hayashi and J.-C. Saut, Global existence of small solutions to the Davey-Stewartson and the Ishimori systems, Diff. Integral Eqs. **8** (1995), 1657-1675.

[23] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. Henri Poincaré, Phys. Théor. **46** (1987), 113-129.

[24] T. Kato, Nonlinear Schrödinger equations, in "Schrödinger Operators" (H. Holden and A. Jensen, Eds.), Lecture Notes in Physics **345**, Springer, 1989.

[25] C.E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle, Comm. Pure Appl. Math. **46** (1993), 527-620.

[26] C.E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, Invent. Math. **134** (1998), 489-545.

[27] H. Komatsu, A general interpolation theorem of Marcinkiewicz type, Tôhoku Math. J. **33** (1981), 383-393.

[28] F. Linares and G. Ponce, On the Davey-Stewartson systems, Ann. Inst. Henri Poincaré, Anal. non linéaire **10** (1993), 523-548.

[29] K. Nakanishi, Asymptotically free solutions for short range nonlinear Schrödinger equations, SIAM J. Math. Anal. **32** (2001), 1256-1271.

[30] K. Nakanishi and T. Ozawa, Remarks on small data scattering for nonlinear Schrödinger equations, NoDEA **9** (2002), 45-68.

[31] T. Ozawa, Long range scattering for nonlinear Schrödinger equations in one space dimension, Commun. Math. Phys. **139** (1991), 479-493.

[32] T. Ozawa and Y. Yamazaki, Smoothing effect and large time behavior of solutions to Schrödinger equations with nonlinearity of integral type, preprint.

[33] W. A. Strauss, Mathematical aspects of classical nonlinear field equations, in "Nonlinear Problem in Theoretical Physics," "Lecture Notes in Physics **98**, Springer, 1979.

[34] Y. Tsutusmi, L^2-solutions for nonlinear Schrödinger equations and nonlinear groups, Funkcialaj Ekvacioj **30** (1987), 115-125.

[35] K. Yajima, Existence of solutions for Schrödinger evolution equations, Commun. Math. Pnys. **110** (1987), 415-426.

[36] W.P. Ziemer, "Weakly Differentialble Functions," GTM **120**, Springer, 1989.