Propagation of the Homogeneous Wave Front Sets for Schrödinger Equations

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For $a(x,\xi) \in C^\infty(\mathbb{R}^{2d})$, we denote the Weyl quantization by:

$$a^w(x,D_x)\varphi(x) = \frac{1}{(2\pi)^d} \iint e^{i(x-y)\xi} a\left(\frac{x+y}{2},\xi\right) \varphi(y)dyd\xi$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Definition 1. Let $u \in \mathcal{S}'(\mathbb{R}^d)$. (x_0,ξ_0) is not in the homogeneous wave front set (HWF set) of u if there exists $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2d})$ such that $a(x_0,\xi_0) \neq 0$ and

$$\|a^w(hx,hD_x)u\| \leq C_N h^N \quad (h \to 0)$$

for any N, where $\|\cdot\|$ denotes the L^2-norm. We denote $(x_0,\xi_0) \notin HWF(u)$. The homogeneous wave front set $HWF(u)$ is defined as the complement in $\mathbb{R}^{2d} \setminus \{0\}$.

Remark. (1) $HWF(u)$ is a conic set in $\mathbb{R}^{2d}$.
(2) The usual wave front set is defined as follows: $(x_0,\xi_0) \notin WF(u)$ if there exists $a(x,\xi) \in C_0^\infty(\mathbb{R}^{2d})$ such that $a(x_0,\xi_0) \neq 0$ and

$$\|a^w(x,hD_x)u\| \leq C_N h^N \quad (h \to 0)$$

for any N. Thus $HWF(u)$ is analogous to $WF(u)$ but homogeneous in (x,ξ).

We consider Schrödinger equation:

$$\frac{\partial}{\partial t}u(t,x) = -iHu(t,x), \quad (t \in \mathbb{R})$$

for $u(t,\cdot) \in L^2(\mathbb{R}^d)$, where

$$H = H_0 + V(x), \quad H_0 = \sum_{j,k=1}^d D_j a_{jk}(x) D_k, \quad D_j = -i\partial/\partial x_j.$$

Assumption A. $a_{jk}(x)$ and $V(x)$ are smooth, and $(a_{jk}(x))$ is positive definite for each x. Moreover, there exist $\mu > 0$ and $\nu < 2$ such that for any $\alpha \in \mathbb{Z}_+^d$,

$$|\partial_x^\alpha(a_{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|},$$
$$|\partial_x V(x)| \leq C'_\alpha \langle x\rangle^{\nu-|\alpha|} \quad (x \in \mathbb{R}^d).$$

We denote

$$h_0(x,\xi) = \sum_{j,k=1}^d a_{jk}(x)\xi_j\xi_k$$

be the symbol of H_0. For $(x,\xi)\in\mathbb{R}^{2d}$, we write the solution to the Hamilton equation:

$$\frac{d}{dt}y(t)=\frac{\partial h_0}{\partial\xi}(y(t),\eta(t)),\qquad \frac{d}{dt}\eta(t)=-\frac{\partial h_0}{\partial x}(y(t),\eta(t))$$

with the initial condition $y(0)=x, \eta(0)=y$ by $y(t,x,\xi)$ and $\eta(t,x,\xi)$.

We say (x,ξ) is *forward nontrapping* with respect to the evolution if $|y(t,x,\xi)|\to\infty$ as $t\to+\infty$. Then it is well-known that there exists an asymptotic momentum:

$$\xi_+=\lim_{t\to+\infty}\eta(t,x,\xi).$$

Theorem 1. *Suppose (x_0,ξ_0) is forward nontrapping, and let ξ_+ be the asymptotic momentum. Let $t_0>0$ and let $u(t)=e^{-itH}u(0)$. If $(t_0\xi_+,\xi_+)\notin HWF(u(t_0))$, then $(x_0,\xi_0)\notin WF(u(0))$. In other words, if $(x_0,\xi_0)\in WF(u(0))$, then $(t_0\xi_+,\xi_+)\in HWF(u(t_0))$.*

Remark. This result is similar to results by Craig-Kappeler-Strauss [CKS]. Also, it is related to works by Doi [Doi], Wunsch, Robbiano-Zuilly (analytic singularity).

Corollary 2. *Suppose (x_0,ξ_0) is forward nontrapping, and let ξ_+ be the asymptotic momentum. Let $s>0$. If $u(t)$ decays rapidly in a conic neighborhood of ξ_+ for some $t>0$, then $(x_0,\xi_0)\notin WF(u(0))$.*

Idea of Proof. We construct an operator $F(t)$ with a symbol: $\varphi(h;t,x,\xi)$ so that

(i) $\varphi(h;t,\cdot,\cdot)\in C_0^\infty(\mathbb{R}^{2d})$ for each $t\geq 0$, $h>0$;

(ii) $\varphi(h;0,x,\xi)=f(x,h\xi)$ with some $f\in C_0^\infty$ such that $f(x_0,\xi_0)>0$;

(iii) $\varphi(h;t_0,\cdot,\cdot)$ is supported in a small conic neighborhood of $(t_0\xi_+,\xi_+)$;

(iv) The Heisenberg derivative of $F(t)$ satisfies:

$$\delta F(t):=\frac{\partial}{\partial t}F(t)+i[H,F(t)]\geq -C_Nh^N$$

for any N in the operator sense.

If we can find such $F(t)=\varphi^w(h;t,x,D_x)$, then Theorem 8 is proved as follows: Suppose $(t_0\xi_+,\xi_+)\notin HWF(u(t_0))$. Then by (4), we have

$$||F(t_0)u(t_0)||=O(h^N)\quad(\forall N).$$

(In fact, we construct $F(t)$ so that this holds.) By (5), we learn

$$\frac{d}{dt}\langle u(t),F(t)u(t)\rangle=\langle u(t),\delta F(t)u(t)\rangle\geq -C_Nh^N,\quad(0\leq t\leq t_0)$$

with any N. Combining them, we obtain

$$\begin{aligned}\langle u(0),F(0)u(0)\rangle&=\langle u(t_0),F(t_0)u(t_0)\rangle-\int_0^{t_0}\langle u(t),\delta F(t)u(t)\rangle dt\\&=O(h^N)+C_Nh^Nt_0=O(h^N)\quad(\forall N).\end{aligned}$$

By (ii), $F(0)=f^w(x,hD_x)$ with $f(x_0,\xi_0)\neq 0$, and this implies $(x_0,\xi_0)\notin WF(u(0))$. □

Construction of the Symbol $\varphi(h;t,x,\xi)$:

We construct $\psi(t,x,\xi)$ so that the Lagrange derivative:

$$\frac{D}{Dt}\,\psi(t,x,\xi) := \frac{\partial\psi}{\partial t} + \frac{\partial h_0}{\partial\xi}\frac{\partial\psi}{\partial x} - \frac{\partial h_0}{\partial x}\frac{\partial\psi}{\partial\xi} \geq 0;$$

$\psi(0,x_0,\xi_0) > 0$; and supported in a small neighborhood of $(y(t,x_0,\xi_0),\eta(t,x_0,\xi_0))$.

Then we set

$$\varphi_0(h;t,x,\xi) = \psi(h^{-1}t, x, h\xi).$$

For $k \geq 1$, we set

$$\varphi_k(h;t,x,\xi) = h^{\varepsilon(k-1)}t\ C_k\ \psi(h^{-1}t, x/\lambda_k, h\xi/\lambda_k)$$

where $\varepsilon = 2-\nu > 0$, $1 < \lambda_1 < \lambda_2 < \cdots < 2$, and C_k are suitable constants. We define

$$\varphi(h;t,x,\xi) \sim \sum_{j=0}^{\infty}\varphi_j(h;t,x,\xi)$$

in the sense of an asymptotic sum with respect to h, and set

$$F(t) = \varphi^w(h;t,x,D_x).$$

Then we can show $F(t)$ satisfies the required properties (i)–(iv). □

References: [CKS] Craig, W., Kappeler, T., Strauss, W.: Microlocal disipertive smoothing for the Schrödinger equation. Comm. Pure Appl. Math. **48**, 769–860 (1996).

[Doi] Doi, S: Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow. Math. Ann. **318**, 355–389 (2000).