Local C-cosine families and local n-times integrated cosine families

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Here we are concerned with local C-cosine families and local n-times integrated cosine families. The characterization of each is mentioned through their infinitesimal generators. The relationship of local C-cosine families, local integrated cosine families and the abstract Cauchy problem is also mentioned.

I. Local C-cosine families.

Let X be a Banach space. Here C is a bounded linear operator in X and the range $R(C)$ is dense in X. We restrict our argument that C is an injective operator. A local C-cosine family called **nondegenerate** if condition $C(t)x = 0$ for any $t \in (0, T)$ implies $x = 0$.

DEFINITION 1.

A family of bounded linear operators $\{C(t) : |t| < T\}$ is called a **local C-cosine family** on X if

(i) $[C(t+s) + C(t-s)]C = 2C(t)C(s)$ for any $t, s, t \pm s \in (-T, T)$,

(ii) $C(0) = C$,

(iii) $C(\cdot)$ is strongly continuous on $(-T, T)$.

DEFINITION 2.

The **infinitesimal generator** of $\{C(t) : |t| < T\}$ is defined as the limit

$$G_0 x := \lim_{h \to 0+} \frac{2}{h^2}[C^{-1}C(h)x - x], \quad x \in D(G_0)$$

with a natural domain $D(G_0) := \{x \in R(C) : \exists \lim_{h\to 0+} \frac{2}{h^2}[C^{-1}C(h)x - x]\}$.

The operator $G = \overline{G_0}$ is said to be the **complete infinitesimal generator** of a local C-cosine family $C(\cdot)$.

We associate with $C(\cdot)$ the local C-sine family by the formula

$$S(t)x = \int_0^t C(s)x ds, \; x \in X.$$

Next, let $\tau \in (0, T)$. Put

$$L_\tau(\lambda)x := \int_0^\tau e^{-\lambda t} S(t)x dt, x \in X,$$

this is so-called the "**local Laplace transform**" of $S(\cdot)$.

Then

$$\lambda L_\tau(\lambda) = \int_0^\tau e^{-\lambda t} C(t)x dt - e^{-\lambda\tau} S(\tau)x.$$

and $L_\tau(\lambda)x \in D(G)$ with

$$(\lambda^2 - G)L_\tau(\lambda)x = Cx - e^{-\lambda\tau}[C(\tau)x + \lambda S(\tau)x],$$

for any $x \in X$.

We set $V_\tau(\lambda)x := -e^{-\lambda\tau}[C(\tau)x + \lambda S(\tau)x]$.

DEFINITION 3.

Let A be a closed linear operator in X. Let $a \in R$ and $\tau \in (0, T)$. A family of operators $\{L_\tau(\lambda) : \lambda > a\} \subset B(X)$ is called **the asymptotic C-resolvent** of A if the following conditions are satisfied:

(i) $L_\tau(\cdot)x \in C^\infty((a, \infty); X)$ for any $x \in X$,

(ii) $L_\tau(\lambda)L_\tau(\mu) = L_\tau(\mu)L_\tau(\lambda)$,

(iii) $L_\tau(\lambda)x \in D(A)$ for any $x \in X$ and

$$(\lambda^2 - A)L_\tau(\lambda)x = Cx + V_\tau(\lambda)x,$$

where $V_\tau(\lambda) \in C^\infty((a,\infty);X)$, and

$$\|\frac{d^n}{d\lambda^n}V_\tau(\lambda)x\| \leq M_\tau \lambda \tau^{n+1} e^{-\lambda\tau}\|x\|,$$

(iv) $AL_\tau(\lambda)x = L_\tau(\lambda)Ax$ for any $x \in D(A)$.

Now we can get the following generation theorem of a local C-cosine family.

THEOREM 1.
Let A be a closed linear operator in X. Then A is the complete infinitesimal generator of a local C-cosine family $C(\cdot)$ on X iff

(i) $D(A)$ is dense in X,

(ii) there is the asymptotic C-resolvent $L_\tau(\lambda)$ of operator A such that for $x \in X$

$$\|\frac{d^n}{d\lambda^n}[\lambda L_\tau(\lambda)x]\| \leq M_\tau \frac{n!}{\lambda^{n+1}}\|x\|,$$

with $0 \leq n/\lambda \leq \tau, \lambda > a, n \in N \cup \{0\}$,

(iii) $CD(A)$ is a core for A.

II. The abstract Cauchy problems.
We consider the abstract Cauchy problem on $(-T,T)$ in the form (ACP; T,x,y) :

$$(\frac{d^2}{dt^2})u(t) = Au(t),\ |t| < T, u(0) = x, u'(0) = y.$$

A function $u(\cdot)$ is called a **solution** to (ACP; T,x,y) if
(a) $u(\cdot)$ is twice continuously differentiable in $t \in (-T,T)$, for any $|t| < T$,
(b) $u(t) \in D(A)$ for any $|t| < T$ and $u(\cdot)$ satisfies (ACP; T,x,y).

We denote also (ACP; T,x,y) with $x,y \in CD(A)$ as (ACP;T,CD(A)).

DEFINITION 4.
The Cauchy problem $(ACP;T,CD(A))$ is said to be **well-posed** if for every $x,y \in CD(A)$ there is a unique solution $u(t;x,y)$ to $(ACP;T,x,y)$ such that

$$\|u(t;x,y)\| \leq M(t)(\|C^{-1}x\| + \|C^{-1}y\|)$$

for $|t| < T$ and $x,y \in CD(A)$, where the function $M(t)$ is bounded on every compact subinterval of $(-T,T)$.

THEOREM 2.
Let A be a densely defined closed linear operator in X satisfying
(i) $Cx \in D(A)$ and $ACx = CAx$ for $x \in D(A)$,
(ii) $CD(A)$ is a core for A.

Then the following are equivalent :

(I) The operator A is the complete infinitesimal generator of a local C-cosine family $C(\cdot)$;

(II) $(ACP;T,CD(A))$ is well-posed.
In this case $u(t;x,y) = C^{-1}C(t)x + C^{-1}S(t)y$, $t \in (-T,T)$, is a unique solution for every initial values $x,y \in CD(A)$.

III. Local n-times integrated cosine families.

DEFINITION 5.

Let $n \geq 1$ and $T \in (0, \infty]$. A family of operators $\{U(t) : |t| < T\}$ in $B(X)$ is called a **local n-times integrated (nondegenerate) cosine family** on X if

(1) $U(\cdot)x : (-T, T) \to X$ is continuous for any $x \in X$;

$$(2) \quad 2U(t)U(s)x = \frac{1}{(n-1)!}[\int_t^{t+s}(t+s-r)^{n-1}U(r)xdr - \int_0^s (t+s-r)^{n-1}U(r)xdr]$$

$$+\frac{(-1)^n}{(n-1)!}[\int_t^{t-s}(t-s-r)^{n-1}U(r)xdr - \int_0^{-s}(t-s-r)^{n-1}U(r)xdr]$$

for $|t|, |s|, |t+s|, |t-s| < T$ and $U(0) = 0$;

(3) $U(t)x = 0$ for $t \in (-T, T)$ implies $x = 0$.

DEFINITION 6.

The infinitesimal generator A_0 of a local n-times integrated cosine family $U(\cdot)$ is defined as the limit:

$$A_0 x = \lim_{h \to 0} 2h^{-2}(U^{(n)}(h)x - x)$$

for $x \in D(A_0)$, with domain

$$D(A_0) = \{x \in \bigcup_{0<\delta<T} C^n(\delta) : \exists \lim_{h \to 0} 2h^{-2}(U^{(n)}x - x)\}.$$

where we denote
$C^n(\delta) = \{x \in X : U(\cdot)x : (-\delta, \delta) \to X$ is n-times continuously differentiable$\}$.

The operator $A = \overline{A_0}$ is said to be the **complete infinitesimal generator** of a local n-times integrated cosine family $U(\cdot)$.

THEOREM 3

If A is the complete infinitesimal operator of a local n-times integrated cosine family $U(\cdot)$ and $[(n+1)/2] = m$, then $(ACP; T, D(A^{m+1}))$ is well-posed in the following sense: for every $x, y \in D(A^{m+1})$ there is a unique solution $u(t; x, y)$ of ACP such that

$$\|u(t; x, y)\| \leq M(t)(\|x\|_m + \|y\|_m)$$

for $|t| < T$ and $x, y \in D(A^{m+1})$,
where $M(t)$ is bounded function on every compact subinterval of $(-T, T)$ and $\|z\|_m = \sum_{i=0}^m \|A^i z\|$ for $z \in D(A^m)$.

REMARK. If $n = 2m - 1$, then for $x \in D(A^{m+1})$ and $y \in D(A^m)$ we have a unique solution to ACP.

THEOREM 4

Let A be a density define closed linear operator in X with $\rho(A) \neq \emptyset$ and $n \in N$. Let $c \in \rho(A), T \in (0, \infty]$. Then the following four conditions are equivalent:
(i) A is the complete infinitesimal generator of a local $2m$-times integrated cosine family $U(\cdot)$;
(ii) A is the complete infinitesimal generator of a local C-cosine family $C(\cdot)$ with $C = R(c; A)^m$;
(iii) $\rho(A)$ contains a half line $\{\lambda \in R : \lambda > \omega\}$ for some $\omega > 0$, and for every $\tau \in (0, T)$ there exists a constant $M_\tau > 0$, depending on τ, such that for $x \in D(A^m)$,

$$\|\frac{\lambda^k}{(k-1)!}\frac{d^{k-1}}{d\lambda^{k-1}}[\lambda R(\lambda^2; A)x]\| \leq M_\tau \|x\|_m,$$

for $0 \leq k/\lambda \leq \tau, \lambda > \omega, k \in N$;
(iv) $(ACP; T, D(A^{m+1}))$ is well-posed.
In this case

$$U(t)x = (c - A)^m \int_0^t \int_0^{t_1} ... \int_0^{t_{2m-1}} C(t_{2m})xdt_{2m}...dt_1$$

for $x \in X$ and $|t| < T$.