ON THE EXISTENCE OF SOLUTIONS TO THE BENJAMIN-ONO EQUATION

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1. Intrduction

In this talk, we consider the existence and the uniqueness of solutions to the Benjamin-Ono equation,

$$\begin{cases} \partial_t u + \mathrm{H}\partial_x^2 u + \frac{1}{2}\partial_x(u^2) = 0, & \text{in } \mathbb{R}\times\mathbb{R}, \\ u(0,x) = \phi(x), & \text{in } \mathbb{R}, \end{cases} \tag{1}$$

where H is the Hilbert transform which is defined by

$$\mathrm{H}f = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{x-y}dy = \mathcal{F}^{-1}(-i\,sgn(\xi))\mathcal{F}f,$$

and $\mathcal{F}$ denotes the Fourier transform with respect to x.

Definition 1. Let s_1, s_2, b_1 and b_2 be real numbers. We define a function space $X^{s_1,s_2}_{b_1,b_2}$ as follows;

$$X^{s_1,s_2}_{b_1,b_2} = \left\{ f \in \mathcal{S}'(\mathbb{R}^2); \|f\|_{X^{s_1,s_2}_{b_1,b_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\langle\tau+\xi^2\rangle^{b_1}\langle\tau-\xi^2\rangle^{b_2}\hat{f}(\tau,\xi)\|_{L^2_{\tau,\xi}} < +\infty \right\}. \tag{2}$$

Here $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ and $\hat{f}(\tau,\xi)$ is the Fourier transform of $f(t,x)$ with respect to space and time variables.

We shall find a solution to the associate integral equation of (1),

$$u(t) = U(t)\phi + \int_0^t U(t-s)u(s)\partial_x u(s)ds, \tag{3}$$

instead of the intial value problem (1) directly. Here $U(t)\phi = \exp(-t\mathrm{H}\partial_x^2)\phi = \mathcal{F}^{-1}\exp(-it\xi|\xi|)\mathcal{F}\phi$. Let ψ be a function in $C_0^\infty(\mathbb{R})$ with $0 \le \psi \le 1$, $\psi(t) = 1$ for $|t| \le 1$ and $\psi(t) = 0$ for $|t| \ge 2$. We consider the following integral equation,

$$u(t,x) = \psi(t)U(t)\phi + \psi(t)\int_0^t U(t-s)u(s)\partial_x u(s)ds. \tag{4}$$

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Definition 2. Let s_1 and s_2 be real numbers. Function space $H^{s_1,s_2}(\mathbb{R}) = H^{s_1,s_2}$ is defined by
(5)
$$H^{s_1,s_2}(\mathbb{R}) = \{g(x) \in \mathcal{S}'(\mathbb{R}); \|g\|_{H^{s_1,s_2}} = \|\langle\xi\rangle^{s_1}|\xi|^{s_2}\hat{g}(\xi)\|_{L^2} < +\infty\}.$$

Our main theorem is the following.

Theorem 1. *Let $1/2 < b < 3/4$. Suppose that $\phi \in H^{2b,-1/2}(\mathbb{R})$ and $\|\phi\|_{H^{2b,-1/2}} \ll 1$. Then there exists a unique solution $u(t,x)$ to the integral equation (4) in $X_{b,b}^{0,-1/2}$. Moreover, we have*

$$\|u_1(t,x) - u_2(t,x)\|_{X_{b,b}^{0,-1/2}} \leq C\|\phi_1 - \phi_2\|_{H^{2b,-1/2}},$$

where u_j is a solution to the equation (4) with initial data ϕ_j for $j = 1,2$.

J. Bourgain[2] has shown L^2 local wellposedness for the Korteweg-de Vries equation. Kenig-Ponce-Vega[5] has extended this result to H^s local wellposedness with $s > -3/4$. For the Benjamin-Ono equation, L. Abdelouhab, J. L. Bona, M. Felland and J. C. Saut[1] and Iorio Jr.[3] has shown global wellposedness for $s > 3/2$. Ponce[7] has shown global wellposedness for $s = 3/2$. Recently Koch and Tvetkov[6] has shown local wellposedness for $s > 5/4$. Very recently, C. E. Kenig and K. D. Koenig[4] has shown local wellposedness for $s > 9/8$. T. Tao[8] has shown local and global wellposedness for $s = 1$.

We prove this theorem by combining the following lemmas.

Lemma 1. *For $s_1, s_2, b \in \mathbb{R}$ and $\phi \in \mathcal{S}(\mathbb{R})$, we have*

$$\|\psi(t)U(t)\phi\|_{X_{b,b}^{s_1,s_2}} \leq C\|\phi\|_{H^{s_1+2b,s_2}},$$

Where $\|\phi\|_{H^{2b,-\rho}} = \||\xi|^{-\rho}\langle\xi\rangle^{2b}\hat{\phi}(\xi)\|_{L^2}$.

Lemma 2. *For$s_1, s_2 \in \mathbb{R}, 1/2 < b \geq 1$ and $f(t,x) \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\|\psi(t)\int_0^t U(t-s)f(s,x)ds\|_{X_{b,b}^{s_1,s_2}} \leq C\|f\|_{X_{b-1,b}^{s_1,s_2}} + C\|f\|_{X_{b,b-1}^{s_1,s_2}}.$$

Lemma 3. *For $1/2 < b < 3/4$, the following inequalities hold for $f, g \in \mathcal{S}(\mathbb{R}^2)$:*

$$\|f\partial_x g\|_{X_{b-1,b}^{0,-1/2}} \leq C\|f\|_{X_{b,b-1}^{0,-1/2}}\|g\|_{X_{b,b}^{0,-1/2}}$$
$$\|f\partial_x g\|_{X_{b,b-1}^{b,-1/2}} \leq C\|f\|_{X_{b,b}^{0,-1/2}}\|g\|_{X_{b,b}^{0,-1/2}}.$$

Remark 1. *$H^{2b,-1/2} \approx H^{2b-1/2}$ in high frequency region.*

Remark 2.
$$X_{b,b}^{0,-1/2} \subset C(\mathbb{R}; H^{2b,-1/2}).$$

References

[1] L. Abdelouhab, J. L. Bona, M. Felland and J. C. Saut, *Nonlocal models for nonlinear dispersive waves*, Phys. D **40**(1989), 360–392.

[2] J. Bourgain, *Fourier ristriction phenomena for certain lattice subset and applications to nonlinear evolution equations: PartII the KdV equation*, Geom. and Funct. Anal. **3**(1993), 209–262.

[3] R. J. Iorio Jr., *On the Cauchy problem for the Benjamin-Ono equation*, Comm. Partial Differential Equations **11**(1986), 1031–1081.

[4] C. E. Kenig and K. D. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Reserch Lett. **10**(2003), 879–895.

[5] C. E. Kenig, G. Ponce and L. Vega, *The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices*, Duke Math. J. **71**(1993), 1–21.

[6] H. Koch and N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equations*, Inst. Math. Res. Not. **26**(2003), 1449–1464.

[7] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation*, Differential Integral Equations **4**(1991), 527–542.

[8] T. Tao, *Global well-posedness of the Benjamin-Ono equation in* H^1, Preprint(2003).