Time local well-posedness for Benjamin-Ono equation with large initial data

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We consider the initial value problem for the Benjamin-Ono equation:

$$
\begin{cases} \partial_t u + \mathcal{H}_x \partial_x^2 u + u \partial_x u = 0, & x, t \in \mathbf{R}, \\ u(x,0) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{0.1}
$$

where $\mathcal{H}_x$ denotes the Hilbert transform, i.e., $\mathcal{H}_x = \mathcal{F}^{-1}(-i\xi/|\xi|)\mathcal{F}$. The equation (0.1) arises in the study of long internal gravity waves in deep stratified fluid.

We present the time local well-posedness of (0.1). Namely, we prove the existence, uniqueness of the solution and the continuous dependence on the initial data. There are several known results about this problem. One of their concern is to overcome the regularity loss arising from the nonlinearity. Because of this difficulty, the contraction mapping principle via the associated integral equation does not work as long as we consider the estimates only in the Sobolev space $H_x^{s,0}$, where $H_x^{s,\alpha}$ is defined by

$$
H_x^{s,\alpha} = \{f \in \mathcal{S}'(\mathbf{R}); \|f\|_{H_x^{s,\alpha}} < \infty\}
$$

with $\|f\|_{H_x^{s,\alpha}} = \|\langle x\rangle^\alpha \langle D_x\rangle^s f\|_{L_x^2}$, $\langle x\rangle^\alpha = (1+x^2)^{\alpha/2}$ and $\langle D_x\rangle^s = \mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$. Indeed, Molinet-Saut-Tzvetkov [6] negatively proved the solvability of the integral equation in $H_x^{s,0}$ for any $s \in \mathbf{R}$.

Recently, Koch-Tzvetkov [4] (see also Ponce [7]) have studied the local well-posedness with $s > 5/4$ due to the cut off technique of $\mathcal{F}u(\xi)$. Furthermore, Kenig-Koenig [2] proved the local well-posedness with $s > 9/8$. We remark here that it is possible to minimize the regularity of u_0 by inducing another kind of function space. In fact, Kenig-Ponce-Vega [3] construct a time local solution via the integral equation by applying the smoothing property like

$$
\|D_x \int_0^t V(t-t')F(t')dt'\|_{L_x^\infty(L_T^2)} \le C\|F\|_{L_x^1(L_T^2)},
$$

where $\|u\|_{L_x^p(L_T^r)} = \|(\|u\|_{L^r[0,T]})\|_{L_x^p(\mathbf{R})}$, $D_x = \mathcal{F}^{-1}|\xi|\mathcal{F}$ and $V(t) = \exp(-t\mathcal{H}_x\partial_x^2)$. They obtained the time local well-posedness in $H_x^{s,0}$ $(s > 1)$ for the cubic nonlinearity (Their argument is also applicable to the quadratic case if u_0 satisfies $u_0 \in H_x^{s,0}$ $(s > 1)$ and the additional weight condition). In their result, however, the smallness of the initial data is required. This is because the inclusion $L_x^1(L_T^\infty) \cdot L_x^\infty(L_T^2) \subset L_x^1(L_T^2)$ yields $\|u\|_{L_x^1(L_T^\infty)}$ in the nonlinearity and we can not expect that $\|u\|_{L_x^1(L_T^\infty)} \to 0$ even when $T \to 0$.

Our concern in this talk is to remove this smallness condition of u_0. Before presenting the rough sketch of our idea, we introduce the function space Y_T in which the solution is constructed:

$$Y_T = \{u : [0,T] \times \mathbf{R} \to \mathbf{R}; \|u\|_{Y_T} < \infty\},$$

where $\|u\|_{Y_T} = \|u\|_{L_T^\infty(H_x^{s,0} \cap H_x^{s_1,\alpha_1})} + \|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}u\|_{L_x^{1/\varepsilon}(L_T^2)} + \|\langle D_x\rangle^{\mu}\langle x\rangle^{\alpha_1}u\|_{L_x^2(L_T^\infty)}$ with $\rho, \mu > 0$ sufficiently small and $0 < \varepsilon < \rho$. We first consider the modified equation such that

$$(0.2) \qquad \begin{cases} \partial_t u_\nu + \mathcal{H}_x\partial_x^2 u_\nu + u_\nu\partial_x\eta_\nu * u_\nu = 0, \\ u_\nu(0,x) = u_0(x), \end{cases}$$

where $\eta_\nu(x) = \nu^{-1}\eta(x/\nu)$ with $\eta \in C_0^\infty$, $\int \eta(x)dx = 1$ and $\nu \in (0,1]$. Then, the existence of u_ν in Y_T easily follows and it is continuated as long as $\|u_\nu(t)\|_{H_x^{s,0}\cap H_x^{s_1,\alpha_1}} < \infty$. Note that $\|u_\nu\|_{Y_T}$ is continuous with respect to T. To seek for the a priori estimate of $\|u_\nu\|_{Y_T}$, we deform (0.2). Let $\varphi \in C_0^\infty(\mathbf{R})$ and write $u_\nu\partial_x\eta_\nu * u_\nu = \varphi\partial_x\eta_\nu * u_\nu + (u_\nu - \varphi)\partial_x\eta_\nu * u_\nu$. Note here that, if φ is close to u_0, one can make $u_\nu - \varphi$ sufficiently small when $t \to 0$. To control $\varphi(\partial_x\eta_\nu * u_\nu)$, we employ the gauge transform so that this quantity is, roughly speaking, absorbed in the linear operator. Then, our desired a priori estimate follows via the integral equation. As for the convergence of nonlinearity $u_\nu\partial_x\eta_\nu * u_\nu \to u\partial_x u$, we also consider the estimate of $u_\nu - u_{\nu'}$. Let us now state our main theorem.

Theorem 0.1 *(i) Let* $u_0 \in H_x^{s,0} \cap H_x^{s_1,\alpha_1} \equiv X^s$ *with* $s_1 + \alpha_1 < s$, $1/2 < s_1$ *and* $1/2 < \alpha_1 < 1$. *Then, for some* $T = T(u_0) > 0$, *there exists a unique solution to (0.1) such that* $u \in C([0,T]; X^s) \cap Y_T$.
(ii) Let $u'(t)$ *be the solution to (0.1) with the initial data* u_0' *satisfying* $\|u_0' - u_0\|_{X^s} < \delta$. *If* $\delta > 0$ *is sufficiently small, then there exist some* $T' \in (0,T)$ *and* $C > 0$ *such that*

$$\|u' - u\|_{L_{T'}^\infty(X^s)} \le C\|u_0' - u_0\|_{X^s},$$
$$\|\langle x\rangle^{-\rho}\langle D_x\rangle^{s+1/2}(u' - u)\|_{L_x^{1/\varepsilon}(L_{T'}^2)} \le C\|u_0' - u_0\|_{X^s}.$$

In Theorem 0.1, the conditions on the initial data are determined by the estimate of maximal function, where, we call $\|f(\cdot,x)\|_{L_T^\infty}$ the maximal function of $f(t,x)$. Concretely speaking, the quantity $\|u\|_{L_x^1(L_T^\infty)}$ is bounded by $C(\|u_0\|_{H_x^{s,0}} + \|u_0\|_{H_x^{s_1,\alpha_1}})$.

Remark. Recently, Tao [8] has studied the global well-posedness in $H_x^{1,0}$ but the L^2-stability of the data-to-solution map holds while the initial data belongs to $H_x^{1,0}$, i.e., $\|u'(t) - u(t)\|_{L^2} \le C\|u_0' - u_0\|_{H_x^{1,0}}$.

We also remark that Koch-Tzvetkov [6] negatively proved the strong stability like

$$\|u'(t) - u(t)\|_{H_x^{s,0}} \leq C\|u_0' - u_0\|_{H_x^{s,0}} \quad \text{for } s > 0,$$

if there is no weight condition on u_0 and u_0'. Though our result requires slightly large regularity in comparison with Tao's work, it suggests that the additional weight condition yields the strong stability of the data-to-solution map in the sense that its target space coincides with that of initial data. Recently, Professor Keiichi Kato [1] obtained the similar result via the Fourier restriction method.

References

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