Sobolev Continuity of a Certain Class of Operators in a Slab Domain in $\mathbb{R}^2$

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In this talk, we consider a Dirichlet problem for the Laplacian in a domain with a corner of conical type in $\mathbb{R}^2$. If that domain is a wedge, then some solutions of that problem are constructed by Mellin transformation concretely, see [5].

We deal with that formula of solutions from the point of view of symbolic calculus. We map that domain into $\mathbb{R} \times [0,1]$ by an appropriate coordinate transformation and introduce a new symbol class and corresponding class of operators. We can construct a global parametrix of that problem in $\mathbb{R} \times [0,1]$ for Sobolev class by means of the symbolic calculus of that class in the form of Theorem 6.

We denote $\mathcal{S} = \mathcal{S}(\mathbb{R})$, $H^s = H^s(\mathbb{R})$, $s \in \mathbb{R}$. Let K be a domain in $\mathbb{R}^2$. Then for $f \in C^\infty(\mathbb{R}^2 \times K)$ $\partial^{(\alpha)}_{(\beta)} f$ denotes $\partial_x^{\alpha_1} \partial_\xi^{\alpha_2} \partial_{\mu_0}^{\beta_1} \partial_{\mu_1}^{\beta_2} f(x, \xi, \mu_0, \mu_1)$, $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \in \mathbb{N}_0^2$. We set $K_1 = \{(\mu_0, \mu_1);\, 0 \le \mu_1 \le \mu_0 \le 1\}$ and $K_{-1} = \{(\mu_0, \mu_1);\, 0 \le \mu_0 \le \mu_1 \le 1\}$. For $\mu \in [0,1]$ we set $I_1^{(\mu)} = [0, \mu]$ and $I_{-1}^{(\mu)} = [\mu, 1]$.

Definition 1 (Ueda [3]) Let $m \in \mathbb{R}$, $\lambda > 0$, $0 \le \delta \le \rho \le 1$, $\delta < 1$ and $i = \pm 1$. We denote by $\mathcal{A}^{m,\lambda}_{\rho,\delta,i}$ the set of all $a \in C^\infty(\mathbb{R}^2 \times K_i)$ such that for every $\alpha = (\alpha_1, \alpha_2), \beta \in \mathbb{N}_0^2$ and $0 < \nu < \lambda$

$$\sup_{\mathbb{R}^2 \times K_i} \left| \partial^{(\alpha)}_{(\beta)} a(x, \xi, \mu_0, \mu_1) \right| \langle \xi \rangle^{-m + \rho\alpha_2 - \delta\alpha_1 - |\beta|} \exp\left(\nu |\mu_1 - \mu_0| \langle \xi \rangle\right)$$

is finite.

The next is a symbol class including the symbol class in Definition 1.

Definition 2 (Ueda [4]) Let $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$, $\delta < 1$ and $i = \pm 1$. We denote by $\mathcal{C}^m_{\rho,\delta,i}$ the set of all $c \in C^\infty(\mathbb{R}^2 \times K_i)$ such that for every $\alpha = (\alpha_1, \alpha_2), \beta \in \mathbb{N}_0^2$

$$\sup_{\mathbb{R}^2 \times K_i} \left| \partial^{(\alpha)}_{(\beta)} c(x, \xi, \mu_0, \mu_1) \right| \langle \xi \rangle^{-m + \rho\alpha_2 - \delta\alpha_1 - |\beta|},$$

$$\sup_{\mathbb{R}^2 \times [0,1]} \int_{I_i^{(\mu_0)}} \left| \partial^{(\alpha)}_{(\beta)} c(x, \xi, \mu_0, \mu_1) \right| d\mu_1 \langle \xi \rangle^{-m+1+\rho\alpha_2 - \delta\alpha_1 - |\beta|},$$

$$\sup_{\mathbb{R}^2\times[0,1]}\int_{I_{-i}^{(\mu_1)}}\left|\partial_{(\beta)}^{(\alpha)}c(x,\xi,\mu_0,\mu_1)\right|d\mu_0\langle\xi\rangle^{-m+1+\rho\alpha_2-\delta\alpha_1-|\beta|}$$

are finite respectively.

An analogous result holds over $\mathcal{C}^{\infty}_{\rho,\delta,i}$ to Hörmander's theorem about the asymptotic expansion in $\mathcal{S}^{\infty}_{\rho,\delta}$.

Definition 3 A linear operator C on $C^0([0,1];\mathcal{S})$ is said to belong to the class $\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$, if there exists $c_i\in\mathcal{C}^m_{\rho,\delta,i}$, $i=\pm1$, such that

$$Cw(\cdot,\mu_0)=\sum_{i=\pm1}\int_{I_i^{(\mu_0)}}c_i(X,D_x,\mu_0,\mu_1)w(\cdot,\mu_1)\,d\mu_1. \tag{1}$$

We shall write $C=\mathrm{OP}(c_1,c_{-1})$ for an operator $C\in\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$ given in the form (1).

The following are our main results.

Theorem 4 *The class* $\bigcup_{m\in\mathbb{R}}\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$ *is an algebra in the following sense:*
(i) *If* $C_j\in\mathrm{Op}\,\mathcal{C}^{m_j}_{\rho,\delta}$, $j=1,2$, *then we have* $C_1C_2\in\mathrm{Op}\,\mathcal{C}^{m_1+m_2-1}_{\rho,\delta}$.
(ii) *If* $C\in\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$, *then we have* $C^*\in\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$.

The next is obtained by using Theorem 4 and multiple symbols of pseudo-differential operators studied in [1].

Theorem 5 *Every operator in* $\mathrm{Op}\,\mathcal{C}^m_{\rho,\delta}$ *is continuous from* $H^l((0,1);H^s)$ *to* $H^{\gamma}((0,1);H^{s-m+1-\gamma})$ *for all* $m,s\in\mathbb{R}$, $l\in\mathbb{N}_0$ *and* $\gamma=0,\dots,l+1$.

Let L be a uniformly elliptic differential operator of second order with complex valued $\mathcal{B}^{\infty}$ coefficients in $\mathbb{R}\times[0,1]$ which have proper ellipticity at some point in $\mathbb{R}\times[0,1]$ and $\partial^2_{\mu_0}+2a_1(x,\mu_0)\partial_x\partial_{\mu_0}+a_2(x,\mu_0)\partial_x^2$ be the principal part of L.

We set $\lambda=\sqrt{a_2-a_1^2}$, $\sigma=2^{-1/2}\inf_{\mathbb{R}\times[0,1]}\bigl(\mathrm{Re}\,\lambda-|\,\mathrm{Im}\,a_1|\bigr)$ and set

$$\begin{aligned}t_1(x,\xi,\mu_0,\mu_1)&=\exp\left(\sqrt{-1}\,(\mu_1-\mu_0)a_1(x,\mu_0)\xi\right)\sinh\left(\mu_1\lambda(x,\mu_0)\xi\right)\\&\qquad\times\frac{\sinh\left((\mu_0-1)\lambda(x,\mu_0)\xi\right)}{\sinh\left(\lambda(x,\mu_0)\xi\right)}\left(\lambda(x,\mu_0)\xi\right)^{-1},\\t_{-1}(x,\xi,\mu_0,\mu_1)&=\exp\left(\sqrt{-1}\,(\mu_1-\mu_0)a_1(x,\mu_0)\xi\right)\sinh\left((\mu_1-1)\lambda(x,\mu_0)\xi\right)\\&\qquad\times\frac{\sinh\left(\mu_0\lambda(x,\mu_0)\xi\right)}{\sinh\left(\lambda(x,\mu_0)\xi\right)}\left(\lambda(x,\mu_0)\xi\right)^{-1}.\end{aligned}$$

We have that σ is positive and $t_i\in\mathcal{A}^{-1,\sigma}_{1,0,i}$, $i=\pm1$, see [3]. We put $T=\mathrm{OP}(t_1,t_{-1})$. Then we obtain

Theorem 6 *One can find $F_j \in \operatorname{Op}\mathcal{C}^0_{1,0}$, $j = 1, 2$, which satisfies the following:*

(i) *If $w \in L^2((0,1); H^s)$, then we have*

$$\big(LT(I+F_1) - I\big)w = K_1 w$$

where $K_1 \in \bigcap_{m\in\mathbb{R}} \operatorname{Op}\mathcal{C}^m_{1,0}$.

(ii) *If $w \in H^2((0,1); H^s)$ with $w(\cdot,0) = w(\cdot,1) = 0$, then we have*

$$\big((I+F_2)TL - I\big)w = K_2 w$$

where $K_2 \in \bigcap_{m\in\mathbb{R}} \operatorname{Op}\mathcal{C}^m_{1,0}$.

(iii) *If $w \in L^2((0,1); H^s)$, $w_0, w_1 \in H^{r+3/2}$ and $f \in L^2((0,1); H^r)$ satisfy that*

$$\begin{aligned}&\int_0^1 (w(\cdot,\mu_0), L^*\varphi(\cdot,\mu_0))\, d\mu_0 \\ &\quad = \int_0^1 (f(\cdot,\mu_0), \varphi(\cdot,\mu_0))\, d\mu_0 + \sum_{j=0}^{1} (-1)^{j+1}\big(w_j, a_2(\cdot,j)\partial_{\mu_0}\varphi(\cdot,j)\big)\end{aligned}$$

for all $\varphi \in C^2([0,1]; \mathcal{S})$ with $\varphi(\cdot,0) = \varphi(\cdot,1) = 0$, then we have $\partial_x^\beta \partial_{\mu_0}^\gamma w \in L^2((0,1); H^{r+2-\beta-\gamma})$, $\beta \in \mathbb{N}_0$, $\gamma = 0, 1, 2$, and $w(\cdot,j) = w_j$, $j = 0, 1$.

(iii) of Theorem 6 is proved by using only Theorem 5, (i) and (ii) of Theorem 6 and Riesz's Theorem.

References

[1] H. Kumano-go, *Pseudo-differential operators of multiple symbol and the Calderón-Vaillancourt theorem,* J. Math. Soc. Japan, **27** (1975), 113–120.

[2] B.-W. Schulze, *Pseudo-Differential Operators on Manifolds with Singularities,* North-Holland, Amsterdam, 1991.

[3] H. Ueda, *On an algebra of a certain class of operators in a slab domain in* $\mathbf{R}^2$*,* Ann. Glob. Anal. and Geom., **16** (1998), 477–495.

[4] H. Ueda, *Sobolev Continuity of a Certain Class of Operators in a Slab Domain in* $\mathbb{R}^2$, Preprint, Osaka University, 2004.

[5] A. H. Zemanian, *Generalized Integral Transformations,* Interscience Publishers, New York, 1968.