Weak solutions to the Navier-Stokes-Poisson equation

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In this talk, I am going to talk about the existence of the weak solutions to the Navier-Stokes-Poisson equation. The results in this talk were obtained in a joint work with Takashi SUZUKI (Osaka Univ.). We consider the Navier-Stokes-Poisson equation

$$
\begin{aligned}
&\rho_t + \nabla \cdot (\rho u) = 0 \\
&(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \rho \nabla \Phi + a \nabla \rho^\gamma = \mu \Delta u + (\lambda + \mu) \nabla (\nabla \cdot u) \\
&\Delta \Phi = 4\pi g \left(\rho - \frac{1}{|\Omega|} \int_\Omega \rho \right) \qquad \text{in } \Omega \times (0,T)
\end{aligned} \tag{1}
$$

with the initial-boundary condition

$$
\begin{aligned}
&u = 0, \quad \frac{\partial \Phi}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \\
&\rho|_{t=0} = \rho_0(x), \quad (\rho u)|_{t=0} = q_0(x) \qquad \text{in } \Omega,
\end{aligned} \tag{2}
$$

where $\Omega \subset \mathbf{R}^3$ is a bounded domain with $C^{2,\theta}$ boundary $\partial\Omega$ $(0 < \theta < 1)$, ν the outer normal vector, $\rho = \rho(x,t)$ the density,

$$
u = u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))
$$

the velocity, $\Phi = \Phi(x,t)$ the Newtonian gravitational potential, $\gamma > 1$ the adiabatic constant, $\mu > 0$ and λ the viscosity constants satisfying $\lambda + \frac{2}{3}\mu \geq 0$, $a = e^S$ the constant determined by the entropy S, and $g > 0$ the gravitational constant. Physically, this system describes the motion of compressible viscous isentropic gas flow under the self-gravitational force.

The equation (1) is provided with the properties of the conservation of total mass $M = \int_\Omega \rho$ and the decrease of total energy E;

$$\begin{aligned} E &= \int_\Omega \left(\frac{\rho}{2} |u|^2 + \frac{P}{\gamma - 1} \right) + \frac{g}{2} \int \int_{\Omega \times \Omega} G(x, y) \rho(x) \rho(y) dx dy \\ &= \frac{a}{\gamma - 1} \|\rho\|_\gamma^\gamma + \frac{1}{2} \|\sqrt{\rho} u\|_2^2 - \frac{1}{8\pi g} \|\nabla \Phi\|_2^2, \end{aligned}$$

and here, $P = a\rho^\gamma$ and $G = G(x, y)$ denote the pressure and the Green's function of the Poisson part, respectively, so that $\Phi(x) = g \int_\Omega G(x, y) \rho(y) dy$ if and only if

$$\Delta \Phi = 4\pi g \left(\rho - \frac{1}{|\Omega|} \int_\Omega \rho \right) \quad \text{in } \Omega, \qquad \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{in } \partial\Omega, \qquad \int_\Omega \Phi = 0. \quad (3)$$

Our result on the non-equilibrium state is regarded as the generalization of Feireisl, Nevotný, and Petzeltová [3] concerning the Navier-Stokes equation without the Poisson term; more precisely,

Theorem 1 *Let $T > 0$ and $\gamma > \frac{3}{2}$. Then, given $\rho_0 \in L^\gamma(\Omega)$ and $|q_0^i|^2 / \rho_0 \in L^1(\Omega)$ with $\rho_0 = \rho_0(x) \geq 0$ and $q_0^i(x) = 0$ for x of $\rho_0(x) = 0$, we have a finite energy weak solution ρ, u, Φ to (1) satisfying the following.*

1. *$\rho = \rho(x, t) \geq 0$, $\rho \in L^\infty(0, T; L^\gamma(\Omega))$, $u^i \in L^2(0, T; H_0^1(\Omega))$.*
2. *$E = E(t) \in L^1_{loc}(0, T)$.*
3. *$\frac{dE}{dt} + \mu \|\nabla u\|_2^2 + (\lambda + \mu) \|\nabla \cdot u\|_2^2 \leq 0$ in $\mathcal{D}'(0, T)$.*
4. *The first two equations of (1) hold in $\mathcal{D}'(\Omega \times (0, T))$.*
5. *$\Phi(\cdot, t) = g \int_\Omega G(\cdot, y) \rho(y, t) dy$ for a.e. $t \in (0, T)$.*
6. *The first equation of (1) holds in $\mathcal{D}'(\mathbf{R}^3 \times (0, T))$ if the zero extension is taken outside Ω to ρ, u.*
7. *The first equation of (1) is satisfied in the sense of the renormalized solution, i.e.,*

$$\frac{d}{dt} b(\rho) + \nabla \cdot (b(\rho) u) + (b'(\rho)\rho - b(\rho)) \nabla \cdot u = 0 \quad (4)$$

in $\mathcal{D}'(\Omega \times (0, T))$ for any $b \in C^1(\mathbf{R})$ such that $b'(z) = 0$ if $|z|$ is large.

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