ANALYTIC SOLUTIONS OF NONLINEAR SCHRÖDINGER EQUATIONS

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Consider the Cauchy problem for the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\Delta\psi=\lambda|\psi|^{2\sigma}\psi,\quad \psi(0,x)=\phi(x), \tag{1}$$

where $x\in\mathbb{R}^n$, $\lambda\in\mathbb{C}$, and σ is a positive integer. The existence of analytic solutions to (1) and more general classes of nonlinear Schrödinger equations has been shown under various conditions on the initial data (see.e.g. [2]-[10]). In [9] it is shown under the condition $2/n\le\sigma\le 2/(n-2)$ ($\sigma\ge 2$ if $n=1$) that (1) has global analytic solutions for a class of low energy, exponentially decaying initial data, and for a class of low energy analytic initial data.

Here we give a corresponding result for L^2-data, assuming that

$$\sigma=2/n,\quad n\le 2. \tag{2}$$

By a global L^2-solution of (1), we mean a solution such that

$$\psi\in C([0,\infty),L^2)\cap L^{2\sigma+2}([0,\infty),L^{2\sigma+2}).$$

Theorem. *Assume* (2). *Let* $\phi\in L^2$ *and* Ω *be an open, symmetric, connected neighborhood of* 0 *in* $\mathbb{R}^n$.
(i) *If* $\sup_{y\in\Omega}\|e^{y\cdot x}\phi\|_{L^2}$ *is sufficiently small, then there is a unique global* L^2*-solution* ψ *of* (1) *such that for every* $t>0$, $\psi(t,\cdot)$ *is real analytic and has an analytic continuation to* $\mathbb{R}^n+it\Omega$.
(ii) *If* ϕ *has an analytic continuation* $\widetilde{\phi}$ *to* $\mathbb{R}^n+i\Omega$ *and* $\sup_{y\in\Omega}\|\widetilde{\phi}(\cdot+iy)\|_{L^2}$ *is sufficiently small, then there is a unique global* L^2*-solution* ψ *of* (1) *such that for every* $t\ge 0,\psi(t,\cdot)$ *is real analytic and has an analytic continuation to* $\mathbb{R}^n+i\Omega$.

The proof uses the standard small data strategy [1] and the regularity lemma described below. Let $\mathcal{H}(T_\Omega)$ denote the set of analytic functions on $T_\Omega=\mathbb{R}^n+i\Omega$, and $\mathcal{H}_t(T_\Omega)$ the set of functions f on T_Ω of the form

$$f(x+iy)=\exp\{y\cdot x+\frac{1}{2}ity^2\}\Psi(x+ity),$$

where Ψ is an analytic function on $\mathbb{R}^n+it\Omega$ if $t\neq 0$, and is a measurable function on $\mathbb{R}^n$ if $t=0$. Let $\mathcal{H}_t^p(T_\Omega)$ denote the space of all $f\in\mathcal{H}(T_\Omega)$ for which the norm $\sup_{y\in\Omega}\|f(\cdot+iy)\|_{L^p}$ is finite, and $\mathcal{H}_t^p(T_\Omega)$ the space of all $f\in\mathcal{H}_t(T_\Omega)$ for which the norm $\sup_{y\in\Omega}\|f(\cdot+iy)\|_{L^p}$ is finite. Let

$$U(t)=\exp\{\frac{1}{2}it\Delta\}.$$

Define the action of $U(t)$ on functions f on T_Ω by

$$(U(t)f)(x+iy)=(U(t)f_y)(x),\quad f_y=f(\cdot+iy).$$

Lemma. *Let Ω be an open connected subset of $\mathbb{R}^n$, $n \geq 1$.*
(i) *For all $t, s \in \mathbb{R}$,*

$$U(t)\mathcal{H}^2_s(T_\Omega) = \mathcal{H}^2_{s+t}(T_\Omega),$$
$$U(t)\mathcal{H}^2(T_\Omega) = \mathcal{H}^2(T_\Omega).$$

(ii) *Let $t > 0$, $2 \leq p < \infty$, and F be a map of $[0, t]$ to the set of functions on T_Ω, such that*

$$\int_0^t \left(\int_\Omega \int_{\mathbb{R}^n} |F(s)|^{p'} dxdy \right)^{1/p'} ds < \infty.$$

Let B be an open ball in Ω with $\overline{B} \subset \Omega$. If $F(s) \in \mathcal{H}_s(T_\Omega)$ for a.a. $s \in [0, t]$, *then*

$$U(t - \cdot)F(\cdot) \in L^1([0, t], \mathcal{H}^p_t(T_B)).$$

If $F(s) \in \mathcal{H}(T_\Omega)$ for a.a. $s \in [0, t]$, *then*

$$U(t - \cdot)F(\cdot) \in L^1([0, t], \mathcal{H}^2(T_B)).$$

The part (ii) of the lemma requires an integrability of $F(\cdot)$ rather than a continuity assumed in [9], and allows us to handle the L^2-case.

References

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