MODIFIED WAVE OPERATOR FOR MODIFIED KDV EQUATION

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We study large time asymptotics of small solutions to the modified KdV equation

$$\partial_t u - \frac{1}{3}\partial_x^3 u = \partial_x u^3, \quad (t,x) \in \mathbf{R}\times\mathbf{R}, \tag{1}$$

Notation and function spaces.

$\mathbf{L}^q = \{\phi; \|\phi\|_q < \infty\}$, where $\|\phi\|_q = (\int |\phi(x)|^q dx)^{1/q}$ if $1 \le q < \infty$ and $\|\phi\|_\infty$ = ess.sup$_{x\in\mathbf{R}} |\phi(x)|$ if $p = \infty$. For simplicity we let $\|\phi\| = \|\phi\|_2$. Weighted Sobolev space $\mathbf{H}^{m,s}$ is defined by $\mathbf{H}_p^{m,s} = \{\phi; \|\phi\|_{m,s,p} = \|(1+|x|^2)^{s/2}(1-\partial_x^2)^{m/2}\phi\|_p < \infty\}$, $m, s \in \mathbf{R}$. We use the notation

$$|||v|||_{\infty,a,b} = \|\hat{v}\|_\infty + \|\partial_\xi \hat{v}\|_\infty + \left\| |\xi|^{-a}\, \partial_\xi \hat{v} \right\|_\infty + \left\| |\xi|^{-b}\, \hat{v} \right\|_\infty$$

and $U(t) = \mathcal{F}^{-1}\exp\left(\frac{i}{\rho} t\,|\xi|^2\,\xi\right)\mathcal{F}$.

Modified wave operator was constructed by Ozawa [4] for the cubic nonlinear Schrödinger equations and by H-Ozawa [3] for the derivative nnonlinear Schrödinger equations. However a existence of modified wave operators is not shown for other nonlinear dispersive equations (for example generalized Benjamin-Ono equations with cubic nonlinearities and the mnodified Korteweg-de Vries equation) as far as we know. Our purpose in this talk is to give a result of a modified wave operator for the mnodified Korteweg-de Vries equation under the conditions that the final state is a real valued and odd function.

Theorem 0.1. *We have the asymptotic formulas for large time t*

$$U(t)\phi = v_\phi(t,q) + R_1(t,x), \tag{2}$$

provided that $|||\phi|||_{\infty,\beta,\delta} < \infty$ or $|||\phi|||_{\infty,0,\widetilde{\delta}} < \infty$, where $0 \le \beta < 1$, $0 \le \delta < 2$, $0 \le \widetilde{\delta} < 1$,

$$\begin{aligned} v_\phi(t,q) &= \begin{cases} t^{-\frac{1}{2}}|q|^{-\frac{1}{2}}\left(C_1 e^{i\frac{2}{3}t|q|^3}\widehat{\phi}(q) + \overline{C_1}e^{-i\frac{2}{3}t|q|^3}\widehat{\phi}(-q)\right), x>0 \\ 0, x \le 0 \end{cases} \\ q &= \left(\frac{x}{t}\right)^{\frac{1}{2}}, C_1 = (2i)^{\frac{1}{2}} \end{aligned}$$

and the reminder

$$\begin{aligned} \|R_1(t)\|_\infty &\le C\max\left(t^{-\frac{1}{3}(\beta+2)}, t^{-\frac{1}{3}(\delta+1)}\right)|||\phi|||_{\infty,\beta,\delta}, \\ \|R_1(t)\| &\le C\max\left(t^{-\frac{1}{2}}, t^{-\frac{1}{3}\left(\widetilde{\delta}+\frac{1}{2}\right)}\right)|||\phi|||_{\infty,0,\widetilde{\delta}}. \end{aligned}$$

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Theorem 0.2. *Let* $u_+ \in \mathbf{L}^2$ *be a real valued function,* $\frac{1}{2} < \widetilde{\delta}$ *and* $|||u_+|||_{\infty,0,\widetilde{\delta}} + \|u_+\|_{0,3} + \|u_+\|_{2,0}$ *be sufficiently small, then there exists a unique global solution* $u \in C([1,\infty);\mathbf{L}^2)$ *of (1) such that*

$$\left\| u\left(t\right) - |q|^{-\frac{1}{2}}\, t^{-\frac{1}{2}} 2\,\mathrm{Re}\, C_1 e^{i\frac{2}{3}t|q|^3 + i6\lambda|\widehat{u}_+(q)|^2 \log t}\widehat{u_+}(q)\right\| \le Ct^{-b},$$

where $\frac{1}{3} < b < \frac{1}{2}$.

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