Existence and Nonexistence of Global Solutions in Time for a Reaction-Diffusion System with Inhomogeneous Terms

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次の反応拡散方程式系の初期値問題 (IVPS) を考える。

$$
\text{(IVPS)}\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & \text{in } \mathbf{R}^n \times (0,\infty), \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & \text{in } \mathbf{R}^n \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, & \text{in } \mathbf{R}^n, \\ v(x,0) = v_0(x) \geq 0, & \text{in } \mathbf{R}^n, \end{cases}
$$

但し，$p_1, p_2 \geq 1$, $p_1 p_2 > 1$ とする。

$a \geq 0$ に対し，次の関数空間を導入する:

$$
I^a = \left\{ \xi \in BC(\mathbf{R}^n); \xi(x) \geq 0, \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty \right\},
$$

$$
I_a = \left\{ \xi \in BC(\mathbf{R}^n); \xi(x) \geq 0, \limsup_{m\to\infty} \inf_{x\in\tilde{B}_{r,m}} |x|^a \xi(x) > 0 \right\}。
$$

但し，$\tilde{B}_{r,m} = B_{r|x_m|}(x_m)$ は，x_m を中心とする半径 $r|x_m|$ $(\exists r > 0)$ の球で，$\{x_m\}_{m=1}^{\infty}$ は $0 < |x_m| < |x_{m+1}|$ $(\forall m)$ かつ $\lim_{m\to\infty} |x_m| = \infty$ をみたすとする。

Assumption 1 $K_i(x,t) \geq 0$ $(i = 1,2)$ は連続関数とし，さらに次の仮定をする*:*

(A1) $K_i(x,t) \leq C_U \langle x \rangle^{\sigma_i} (t+1)^{q_i}$ *for any* $x \in \mathbf{R}^n,\ t \geq 0$,

(A2) $K_i(x,t) \geq C_L |x|^{\sigma_i} t^{q_i}$ *for any* $x \in \bigcup_{m=1}^{\infty} \tilde{B}_{r,m},\ t \geq 0$。

但し，$C_U \geq C_L > 0$, $\sigma_i \geq 0$, $q_i \geq 0$ $(i = 1,2)$, $\langle x \rangle = (|x|^2+1)^{1/2}$ とする。

Assumption 2 初期データは，$(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ をみたすとする。但し，

$$
\delta_i = \frac{\sigma_j p_i + \sigma_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1))
$$

である。

$(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$ かつ $K_i(x,t)$ $(i = 1,2)$ が *(A1)* をみたすとき，$\exists T > 0$ に対して $\mathbf{R}^n \times (0,T)$ において (IVPS) の非負の時間局所解 (u,v) が一意的に存在する。

$$
\alpha_i = \frac{(2+\sigma_i+2q_i) + (2+\sigma_j+2q_j)p_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1))
$$

とおくとき，次の主結果を得る:

Theorem 1 (時間大域解の非存在)
Assume that $(u_0, v_0) \in I^{\delta_1} \times I^{\delta_2}$, $(u_0, v_0) \not\equiv 0$, *and that* $K_i(x,t)$ $(i = 1, 2)$ *satisfy (A1) and (A2). Suppose that one of the following three conditions holds;*

(i) $\max\{\alpha_1, \alpha_2\} \geq n$.

(ii) $u_0 \in \tilde{I}_{a_1}$ *with* $a_1 < \alpha_1$ *or* $v_0 \in \tilde{I}_{a_2}$ *with* $a_2 < \alpha_2$.

(iii) $u_0(x)$ *or* $v_0(x) \geq M e^{-\nu_0 |x|^2}$ *for some* $\nu_0 > 0$ *and* $M > 0$ *large enough.*

Then, every solution (u, v) *of* (IVPS) *is not global in time.*

Theorem 2 (時間大域解の存在)
Assume that $\max\{\alpha_1, \alpha_2\} < n$, *and that* $K_i(x,t)$ $(i = 1, 2)$ *satisfy (A1). Suppose that*

$$(u_0, v_0) \in I^{a_1} \times I^{a_2} \text{ with } a_1 > \alpha_1,\ a_2 > \alpha_2,$$

and that $\|\langle\cdot\rangle^{a_1} u_0\|_\infty$ *and* $\|\langle\cdot\rangle^{a_2} v_0\|_\infty$ *are small enough. Then, every solution* (u, v) *of* (IVPS) *is global in time.*

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