Identification of the absent spectral gaps in a class of generalized Kronig-Penney Hamiltonians

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In this talk we study the spectral gaps of the one-dimensional Schrödinger operators with particular periodic point interactions. We fix $\kappa \in (0,\pi) \cup (\pi, 2\pi)$. Let

$$\Gamma_1 = 2\pi\mathbf{Z}, \quad \Gamma_2 = \{\kappa\} + 2\pi\mathbf{Z}, \quad \Gamma = \Gamma_1 \cup \Gamma_2.$$

For $\theta_1, \theta_2 \in [-\pi/2, \pi/2)$ and $A_1, A_2 \in SO(2)\backslash\{\pm I\}$, we define the operator $H = H(A_1, A_2, \theta_1, \theta_2)$ in $L^2(\mathbf{R})$ as follows.

$$(Hy)(x) = -\frac{d^2}{dx^2}y(x), \quad x \in \mathbf{R} \setminus \Gamma,$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbf{R} \setminus \Gamma) \;\middle|\; \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = e^{i\theta_j} A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix}, \quad x \in \Gamma_j, \quad j = 1, 2 \right\}.$$

Since $A_j \in SO(2) \setminus \{\pm I\}$, we can write the elements of A_j as

$$A_j = \begin{pmatrix} \cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j \end{pmatrix}, \quad \alpha_j \in (-\pi, 0) \cup (0, \pi).$$

The operator H is self-adjoint. Since the set $\sigma(H(\theta_1, \theta_2, A_1, A_2))$ is independent of θ_1 and θ_2, we hereafter discuss only the case where

$$\theta_1 = \theta_2 = 0.$$

Next, we define the spectral gaps of H. To this end, we consider the equation

$$\begin{cases} -y''(x,\lambda) = \lambda y(x,\lambda), \quad x \in \mathbf{R} \setminus \Gamma \\ \begin{pmatrix} y(x+0,\lambda) \\ y'(x+0,\lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0,\lambda) \\ y'(x-0,\lambda) \end{pmatrix} \text{ for } x \in \Gamma_j, \ j = 1, 2, \end{cases} \tag{1}$$

where λ is a real parameter. This equation has two solutions $y_1(x,\lambda)$ and $y_2(x,\lambda)$ which are uniquely determined by the initial conditions

$$y_1(+0,\lambda) = 1, \quad y_1'(+0,\lambda) = 0,$$

and

$$y_2(+0,\lambda) = 0, \quad y_2'(+0,\lambda) = 1,$$

respectively. We introduce the discriminant $D(\lambda)$ of the equation (**??**):

$$D(\lambda) = y_1(2\pi + 0, \lambda) + y_2'(2\pi + 0, \lambda). \tag{2}$$

Let λ_j be the $(j+1)$st zero of $D(\cdot)^2 - 4$. Then we have

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots < \lambda_{2k-1} \leq \lambda_{2k} < \cdots \to \infty.$$

We define

$$B_j = [\lambda_{2j-2}, \lambda_{2j-1}], \quad G_j = (\lambda_{2j-1}, \lambda_{2j}).$$

Then we derive

$$\sigma(H) = \bigcup_{j=1}^{\infty} B_j.$$

The open interval G_j is called the j-th gap of the spectrum of H, the closed interval B_j the j-th band. The aim of this study is to determine whether or not the j-th gap is absent for a given $j \in \mathbf{N}$. Throughout this talk we use the notations

$$a \equiv b \ \text{ if } \ a - b \in \pi\mathbf{Z}, \qquad a \not\equiv b \ \text{ if } \ a - b \notin \pi\mathbf{Z}$$

for a, $b \in \mathbf{R}$. For convenience we adopt the following classification of the parameters α_1 and α_2.

$$\begin{aligned} &(\mathrm{I})\ \alpha_1 - \alpha_2 \not\equiv 0, \quad \alpha_1 + \alpha_2 \not\equiv 0. \\ &(\mathrm{II})\ \alpha_1 + \alpha_2 \equiv 0. \\ &(\mathrm{III})\ \alpha_1 - \alpha_2 \equiv 0, \quad \alpha_1 + \alpha_2 \not\equiv 0. \end{aligned}$$

Our main results are the following three theorems.

THEOREM 1. *If the condition* (I) *holds, then*

$$G_j \neq \emptyset \quad \text{for all} \quad j \in \mathbf{N}.$$

THEOREM 2. *Suppose that* (II) *is valid.*
(1) *Let* $\kappa/\pi \notin \mathbf{Q}$. *Then we have*

$$\{j \in \mathbf{N} |\quad G_j = \emptyset\} = \{3\}.$$

(2) *If* $\kappa/2\pi = q/p$, $(p,q) \in \mathbf{N}^2$, $\gcd(p,q) = 1$, *then*

$$\{j \in \mathbf{N} | \quad G_j = \emptyset\} = \{3\} \cup \{pk+1 \ | \ k \in \mathbf{N}\}.$$

THEOREM 3. *Assume that* (III) *is valid. We put* $\eta_j = \pi^2 j^2/4(\pi-\kappa)^2$ *for* $j \in \mathbf{N}$. *Then it holds that*

$$\bigcup_{k=1}^{\infty} B_k \cap B_{k+1} = \left\{\eta_j \Big| -2\Big(\sqrt{\eta_j} + \frac{1}{\sqrt{\eta_j}}\Big)^{-1} \cot \kappa\sqrt{\eta_j} = \tan\alpha_1 \text{ and } j \in \mathbf{N}\right\}.$$

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