Stability of solitary waves for Ostrovsky equations

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この講演は, Yue Liu (University of Texas at Arlington) との共同研究に基づく. In 1978, Ostrovsky [11] derived an equation for weakly nonlinear surface and internal waves in a rotating ocean:

$$\partial_x(\partial_t u + \alpha u \partial_x u + \beta \partial_x^3 u) = \gamma u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where α, β, $\gamma \in \mathbb{R}$, $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In oceanographic applications, the small-scale dispersion (the term with β) appears due to influence of oceanic depth, and the large-scale dispersion (the term with γ) appears due to influence of Earth rotation. In certain situations, in place of (1), the modified Ostrovsky equation arises [12]:

$$\partial_x(\partial_t u + \alpha_1 u^2 \partial_x u + \beta \partial_x^3 u) = \gamma u. \tag{2}$$

Note that when $\gamma = 0$, (1) and (2) are reduced to the KdV equation and the modified KdV equation, respectively.

In this talk, we consider the following Ostrovsky-type equation:

$$\partial_t u + \partial_x^3 u + \partial_x^{-1} u + \partial_x(|u|^{p-1}u) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{3}$$

where $1 < p < \infty$ and for $k \in \mathbb{N}$ the operator ∂_x^{-k} is defined by

$$\partial_x^{-k} v = \mathcal{F}^{-1}[(i\xi)^{-k}\hat{v}(\xi)].$$

We study orbital stability of travelling wave solution $u(t,x) = \varphi_c(x - ct)$ for (3), where $c \in \mathbb{R}$ and φ_c is a ground state of

$$-\partial_x^2 \varphi - \partial_x^{-2} \varphi + c\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}. \tag{4}$$

We define the energy space X by

$$X = \{v \in H^1(\mathbb{R}) : \|v\|_X < \infty\}, \quad \|v\|_X^2 = \int_{\mathbb{R}} (\xi^2 + \xi^{-2})|\hat{v}(\xi)|^2 \, d\xi,$$

and define the energy E and the momentum V by

$$E(v) = \int_{\mathbb{R}} \left\{ \frac{1}{2}|\partial_x v|^2 + \frac{1}{2}|\partial_x^{-1} v|^2 - \frac{1}{p+1}|v|^{p+1} \right\} dx,$$
$$V(v) = \int_{\mathbb{R}} \frac{1}{2}|v|^2 dx.$$

The energy E and the momentum V are conserved quantities of (3). For the well-posedness of [11], see [8].

The equation (4) is considered as the Euler-Lagrange equaiton for the funtional

$$\begin{aligned}S_c(v) &= E(v) + cV(v)\\ &= \int_{\mathbb{R}} \left\{ \frac{1}{2}|\partial_x v|^2 + \frac{1}{2}|\partial_x^{-1} v|^2 + \frac{c}{2}|v|^2 - \frac{1}{p+1}|v|^{p+1} \right\} dx\end{aligned}$$

defined on X. We denote the set of all non-trivial solutions for (4) by

$$\mathcal{S}_c = \{v \in X : S_c'(v) = 0,\ v \neq 0\},$$

and the set of all ground states for (4) by

$$\mathcal{G}_c = \{w \in \mathcal{S}_c : S_c(w) \le S_c(v),\ \forall v \in \mathcal{S}_c\}.$$

Moreover, we put

$$\begin{aligned}K_c(v) &= \int_{\mathbb{R}} \left\{ |\partial_x v|^2 + |\partial_x^{-1} v|^2 + c|v|^2 - |v|^{p+1} \right\} dx,\\ d(c) &= \inf\{S_c(v) : v \in X,\ K_c(v) = 0, v \neq 0\},\\ \mathcal{M}_c &= \{w \in X : S_c(w) = d(c),\ K_c(w) = 0,\ w \neq 0\}.\end{aligned}$$

Note that $K_c(v) = \partial_\lambda S_c(\lambda v)|_{\lambda=1}$ and that if $c > -2$ then $\xi^2 + \xi^{-2} + c > 0$ for all $\xi \in \mathbb{R}$. It can be proved that if $c > -2$, then $\mathcal{M}_c$ is not empty and $\mathcal{G}_c = \mathcal{M}_c$. We remark that the uniqueness of ground states for (4), up to translation, is not known.

We say that the travelling wave solution $\varphi_c(x - ct)$ is orbitally stable in X if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0 - \varphi_c\|_X < \delta$ then the solution $u(t)$ of (3) with $u(0) = u_0$ exists for all $t \in \mathbb{R}$, and satisfies

$$\sup_{t\in\mathbb{R}} \inf_{y\in\mathbb{R}} \|u(t) - \varphi_c(\cdot + y)\|_X < \varepsilon.$$

Otherwise, it is said to be orbitally unstable in X.

Here, we recall some known results for related equations. First, we consider the generalized KdV equation:

$$\partial_t u + \partial_x^3 u + \partial_x(|u|^{p-1}u) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{5}$$

For $c > 0$, (5) has travelling wave solutions $u(t, x) = \psi_c(x - ct)$, where ψ_c is given by

$$\psi_c(x) = \left(\frac{(p+1)c}{2} \right)^{1/(p-1)} \operatorname{sech}^{2/(p-1)} \left(\frac{p-1}{2}\sqrt{c}x \right),$$

which is a positive solution of

$$-\partial_x^2 \varphi + c\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}. \tag{6}$$

Note that $\psi_c(x) = c^{1/(p-1)}\psi_1(\sqrt{c}x)$. It is known that the travelling wave solution $\psi_c(x-ct)$ of (5) is orbitally stable in $H^1(\mathbb{R})$ when $1 < p < 5$ and $c > 0$, and is orbitally unstable in $H^1(\mathbb{R})$ when $p \geq 5$ and $c > 0$ (see Bona, Souganidis and Strauss [2] for the case $p \neq 5$, and Martel and Merle [10] for $p = 5$). The same result holds for the nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{7}$$

That is, the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (7) is orbitally stable in $H^1(\mathbb{R})$ when $1 < p < 5$ and $\omega > 0$, and is orbitally unstable in $H^1(\mathbb{R})$ when $p \geq 5$ and $\omega > 0$ (see [1] and [3]).

Next, we consider nonlinear Schrödinger equation with a harmonic potential:

$$i\partial_t u + \partial_x^2 u - |x|^2 u + |u|^{p-1}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{8}$$

The energy space for (8) is given by

$$\Sigma = \{v \in H^1(\mathbb{R}) : xv \in L^2(\mathbb{R})\}.$$

Note that $\lambda_1 = 1$ is the first eigenvalue of the operator $-\partial_x^2 + |x|^2$ on $L^2(\mathbb{R})$. For any $\omega > -\lambda_1$, there exists a ground state ϕ_ω of

$$-\partial_x^2\phi + |x|^2\phi + \omega\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}. \tag{9}$$

The following results were obtained by Fukuizumi and Ohta [5, 6]. When $1 < p < \infty$, there exists $\omega_1 > -\lambda_1$ such that $e^{i\omega t}\phi_\omega(x)$ is orbitally stable in Σ for all $\omega \in (-\lambda_1, \omega_1)$. When $1 < p < 5$, there exists $\omega_2 > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is orbitally stable in Σ for all $\omega \in (\omega_2, \infty)$. When $p > 5$, there exists $\omega_3 > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is orbitally unstable in Σ for all $\omega \in (\omega_3, \infty)$. Moreover, Fukuizumi [4] proved that when $p = 5$, there exists $\omega_4 > 0$ such that $e^{i\omega t}\phi_\omega(x)$ is orbitally stable in Σ for all $\omega \in (\omega_4, \infty)$.

The proof for the case where $p \neq 5$ and ω is large is based on the the following fact. For $\omega > 0$, we define $\tilde{\phi}_\omega$ by

$$\phi_\omega(x) = \omega^{1/(p-1)}\tilde{\phi}_\omega(\sqrt{\omega}x).$$

Then, for any sequence $\{\omega_j\}$ satisfying $\omega_j \to \infty$, there exists a sequence $\{y_j\}$ of $\mathbb{R}$ such that $\{\tilde{\phi}_{\omega_j}(\cdot + y_j)\}$ has a subsequence that converges to ψ_1 in $H^1(\mathbb{R})$.

Recently, Levandosky and Liu [7] studied orbital stability of $\varphi_c(x-ct)$ for (3). By a similar approach to that used by [6] for (8), they proved that when $p > 5$, there exists $c_3 > 0$ such that $\varphi_c(x-ct)$ is orbitally unstable in X for all $c \in (c_3, \infty)$. See also [9].

The approach in [5] is applicable to the case where $1 < p < 5$ and c is large. That is, we have the following result, which is the main result in this talk.

Main Result (Y. Liu and M. O.) Let $1 < p < 5$ and $\varphi_c \in \mathcal{G}_c$. Then, there exists $c_2 > 0$ such that the travelling wave solution $\varphi_c(x-ct)$ of (3) is orbitally stable in X for all $c \in (c_2, \infty)$.

References

[1] H. Berestycki and T. Cazenave, Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, C. R. Acad. Sci. Paris. **293** (1981) 489–492.

[2] J. Bona, P. Souganidis and W. Strauss, Stability and instability of solitary waves of Korteweg-de Vries type, Proc. Roy. Soc. London, Ser. A **411** (1987) 395–412.

[3] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. **85** (1982) 549–561.

[4] R. Fukuizumi, Stability of standing waves for nonlinear Schrödinger equations with critical power nonlinearity and potentials, Adv. Differential Equations **10** (2005) 259–276.

[5] R. Fukuizumi and M. Ohta, Stability of standing waves for nonlinear Schrödinger equations with potentials, Differential Integral Equations **16** (2003) 111–128.

[6] R. Fukuizumi and M. Ohta, Instability of standing waves for nonlinear Schrödinger equations with potentials, Differential Integral Equations **16** (2003) 691–706.

[7] S. Levandosky and Y. Liu, Stability of solitary waves of a generalized Ostrovsky equation, Preprint.

[8] F. Linares and A. Milanés, Local and global well-posedness for the Ostrovsky equation J. Differential Equations **222** (2006) 325–340.

[9] Y. Liu and V. Varlamov, Stability of solitary waves and weak rotation limit for the Ostrovsky equation, J. Differential Equations **203** (2004) 159–183.

[10] Y. Martel and F. Merle, Instability of solitons for the critical generalized Korteweg-de Vries equation, Geom. Funct. Anal. **11** (2001) 74–123.

[11] L. Ostrovsky, Nonlinear internal waves in a rotating ocean, Okeanologiya **18** (1978) 181–191.

[12] L. Ostrovsky and Y. Stepanyants, Do internal solitons exists in the ocean?, Rev. Geophys. **27** (1989) 293–310.