Analysis of the embedding properties for Weighted Sobolev spaces in unbounded domains

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In this paper we consider some embedding properties for Weighted Sobolev spaces in $\mathbf{R}^N$ with $N \geq 1$. We are concerned with these functional spaces which have exponentially growing functions as weight functions.

Let $\theta(x) \in C^1(\mathbf{R}^N)$ and $p > 1$. Set $\|\cdot\|_{p,\theta}$ semi-norm as following;

$$\|u\|_{p,\theta} = \left(\int_{\mathbf{R}^N} e^{p\theta(x)} |u(x)|^p dx \right)^{1/p}.$$

Corresponding to the above semi-norm, we define $L^p(\theta, \mathbf{R}^N) := \{u \in L^1_{loc}(\mathbf{R}^N); \|u\|_{p,\theta} < +\infty\}$ as a functional space with weight function $e^{p\theta}$. Throughout this paper, we will focus on the case that $\theta(x)$ is bounded below in $\mathbf{R}^N$. In such a situation we can choose $\theta(x)$ as a non-negative function without less of generality. If there exists $c_1 > 0$ satisfying $\theta(x) \geq -c_1$ for all $x \in \mathbf{R}^N$, then we can easily see that $\hat{\theta}(x) := \theta(x) + c_1 \geq 0$ satisfies $\|u\|_{p,\hat{\theta}} = e^{c_1}\|u\|_{p,\theta}$ for all $u \in L^p(\theta, \mathbf{R}^N)$. Thus we can show that these norms $\|\cdot\|_{p,\theta}$ and $\|\cdot\|_{p,\hat{\theta}}$ are equivalent to each other and this is the reason why we can choose θ as a non-negative function. Due to the positivity of this weight function, it is obvious that $L^p(\theta, \mathbf{R}^N)$ is a subspace of $L^p(\mathbf{R}^N)$. Moreover it is easily seen that $\|\cdot\|_{p,\theta}$ is a norm corresponding to $L^p(\theta, \mathbf{R}^N)$.

We also define $D^{1,p}(\theta, \mathbf{R}^N)$ and $W^{1,p}(\theta, \mathbf{R}^N)$ are completions of $C_0^1(\mathbf{R}^N)$ with respect to the norms $\|v\| := \|\nabla v\|_{p,\theta}$ and $\|w\|_{1,p,\theta} := (\|w\|^p_{p,\theta} + \|\nabla w\|^p_{p,\theta})^{1/p}$, respectively. We can also show that $D^{1,p}(\theta, \mathbf{R}^N)$ and $W^{1,p}(\theta, \mathbf{R}^N)$ are both Banach spaces because of $\theta \geq 0$.

In the case $p = 2$, Escobedo-Kavian [**?**] have shown some embedding properties for $W^{1,2}(\theta, \mathbf{R}^N) \subset L^q(\theta, \mathbf{R}^N)$ with $2 \leq q \leq 2N/(N-2)$. They have shown the continuity and compactness of the above embeddings under some conditions on the non-negative function $\theta(x) \in C^2(\mathbf{R}^N)$. They claimed that if $\eta(\theta)(x) := \Delta\theta(x) + |\nabla\theta(x)|^2$ tends to infinity

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as $|x| \to \infty$, then the embedding $W^{1,2}(\theta, \mathbf{R}^N) \subset L^2(\theta, \mathbf{R}^N)$ is compact for $N \geq 3$. They have also studied the continuity and compactness of the embeddings from $W^{1,2}(\theta, \mathbf{R}^N)$ into $L^q(\theta, \mathbf{R}^N)$ under some suitable conditions on $\eta(\theta)$. These functional spaces have an important role to show the existence of decaying solutions for some semilinear elliptic problems which arise in the analysis of semilinear heat equations. We also refer Kawashima [**?**] and Muramoto-Naito-Yoshida [**?**]. They also discuss such kind of functional spaces with special case $\theta(x) = |x|^2$. In such a situation, a simple calculation yields that $\eta(\theta)(x)$ tends to infinity as $|x| \to \infty$.

On the basis of their results, we will focus on the embeddings $D^{1,p}(\theta, \mathbf{R}^N) \subset L^q(\theta, \mathbf{R}^N)$ and $W^{1,p}(\theta, \mathbf{R}^N) \subset L^q(\theta, \mathbf{R}^N)$ with $p, q > 1$ and $q \leq p^* := Np/(N-p)$. In [**?**], we have studied the embedding continuity and compactness under some conditions on $\theta(x) \in C^2(\mathbf{R}^N)$. Our strategy is to find the sufficient conditions on $\theta(x) \in C^1(\mathbf{R}^N)$ and relax them in showing some suitable properties of the above embeddings. We will generalize their arguments for these kind of embeddings.

Before stating our results, we will introduce an auxiliary function which corresponds to $\theta(x)$. Choose $\theta(x) \in C^1(\mathbf{R}^N)$ and fixed. For every $\vec{h} \in (C^1(\mathbf{R}^N))^N$ and $k > 0$, we define

$$G(\vec{h}, k)(x) := p\nabla\theta \cdot \vec{h}(x) + \mathrm{div}\vec{h}(x) - (p-1)k^{-1}|\vec{h}(x)|^{p/(p-1)} \quad \text{for all } x \in \mathbf{R}^N$$

It is easily seen from the above definition that function $G(\vec{h}, k)(x)$ is continuous with respect to $x \in \mathbf{R}^N$. We also note that if $k_1 \geq k_2$

$$G(\vec{h}, k_1)(x) \geq G(\vec{h}, k_2)(x) \quad \text{for all } x \in \mathbf{R}^N$$

for every $\vec{h} \in (C^1(\mathbf{R}^N))^N$.

We assume the following two conditions on $\theta(x)$; there exist $\vec{h}_0 \in (C^1(\mathbf{R}^N))^N$ and $k_0 > 0$ such that

$$\begin{array}{ll} (\theta.1)_{C_1} & G(\vec{h}_0, k_0)(x) \to C_1 \quad \text{as } |x| \to \infty, \\ (\theta.2) & G(\vec{h}_0, k_0)(x) \geq C_2 |\nabla\theta(x)|^p \quad \text{for all } x \in \mathbf{R}^N \end{array}$$

where C_1 and C_2 are positive constants. We simply write $(\theta.1)_\infty$ if we consider the case $C_1 = +\infty$.

From these assumptions, we can obtain the following theorems.

Theorem 1. *Let $p > 1$ and $\theta(x) \in C^1(\mathbf{R}^N)$. Assume $(\theta.1)_{C_1}$ and $(\theta.2)$ with $C_1 < +\infty$. Then the embedding $D^{1,p}(\theta, \mathbf{R}^N) \subset L^p(\theta, \mathbf{R}^N)$ is continuous.*

Theorem 2. *Let $p > 1$ and $\theta(x) \in C^1(\mathbf{R}^N)$. Assume $(\theta.1)_\infty$ and $(\theta.2)$. Then the embedding $D^{1,p}(\theta, \mathbf{R}^N) \subset L^p(\theta, \mathbf{R}^N)$ is continuous and compact.*

Furthermore, we can also obtain the similar embedding properties with $L^{p,\beta}(\theta,\mathbf{R}^N)$ as

$$L^{p,\beta}(\theta,\mathbf{R}^N) := \left\{ u \in L^1_{loc}(\mathbf{R}^N) \Big| \int_{\mathbf{R}^N} e^{p\theta(x)}(1+|\nabla\theta(x)|^p)^\beta |u(x)|^p dx < +\infty \right\},$$

where β is a real number. Corresponding to the above space, we define

$$|[u]|_{p,\beta,\theta} := \left(\int_{\mathbf{R}^N} e^{p\theta}(1+|\nabla\theta|^p)^\beta |u|^p dx \right)^{1/p}$$

as a semi-norm. We note that $|[\cdot]|_{p,\beta,\theta}$ is a norm if β is non-negative.

Theorem 3. *Let $p > 1$ and $\theta(x) \in C^1(\mathbf{R}^N)$. Assume $(\theta.1)_{C_1}$ and $(\theta.2)$ with $C_1 < +\infty$. Then the embedding $D^{1,p}(\theta,\mathbf{R}^N) \subset L^{p,\beta}(\theta,\mathbf{R}^N)$ is continuous for any $\beta \leq 1$.*

Theorem 4. *Let $p > 1$ and $\theta(x) \in C^1(\mathbf{R}^N)$. Assume $(\theta.1)_\infty$ and $(\theta.2)$. Then the embedding $D^{1,p}(\theta,\mathbf{R}^N) \subset L^{p,\beta}(\theta,\mathbf{R}^N)$ is continuous for any $\beta \leq 1$. Furthermore, the above embedding is also compact for any $\beta < 1$.*

We can easily verify that $G(\nabla\theta,1)(x) = \Delta\theta(x) + |\nabla\theta(x)|^2 = \eta(\theta)(x)$ for every $\theta(x) \in C^2(\mathbf{R}^N)$ if $p = 2$. So Theorem 1.2 assures that $W^{1,2}(\theta,\mathbf{R}^N) \subset L^2(\theta,\mathbf{R}^N)$ is compact under the assumptions given in [**?**]. Consequently one can see that these theorems are generalization of their arguments.

References

[1] M. Escobedo and O. Kavian, *Variational problems related to self-similar solutions of the heat equation,* Nonlinear Anal. **11** (1987), 1103-1133.

[2] S. Kawashima, *Self-similar solutions of a convection-diffusion equation,* Lecture Notes Numer. Appl. Anal. **12** (1993), 123-136.

[3] N. Muramoto, Y. Naito and K. Yoshida, *Existence of self-similar solutions to a parabolic system modelling chemotaxis,* Japan J. Indust. Appl. Math. **17** (2000), 427-451.

[4] H. Ohya, *Exponentially decaying solutions of quasilinear elliptic equations in $\mathbf{R}^N$,* Adv. Math. Sci. Appl. **13** (2003) 287-299.