On Sobolev's original polynomial projection and the so-called Bramble-Hilbert Lemma

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1 Introduction

S. L. Sobolev, in his classical works ([1],[2]), introduced and discussed the spaces $W_p^{(m)}(\Omega)$ of generalized functions on a domain $\Omega \subset \mathbb{R}^n$, now called Sobolev spaces.

In particular, Sobolev carried out a very detailed analysis when Ω is *bounded* and *star-shaped* with respect to a closed *ball* K inside it. He considered the following quotient space

$$L_p^{(m)}(\Omega) = W_p^{(m)}(\Omega)/\sim$$

by the equivalence relation

$$u \sim v \iff \partial^\alpha u(x) = \partial^\alpha v(x),\ |\alpha| = m \quad (\text{a.e.} x \in \Omega)$$

for $u, v \in W_p^{(m)}(\Omega)$. On the other hand, let

$$\mathbb{P}^{m-1} = \left\{ \sum_{|\alpha| \leqq m-1} a_\alpha\, x^\alpha \right\}$$

be the space of polynomial functions of degree $\leqq m-1$. Then he showed the Banach space isomorphism

$$W_p^{(m)}(\Omega) = \mathbb{P}^{m-1} \oplus L_p^{(m)}(\Omega)$$

using an explicitly constructed *projection* operator

$$\Pi:\ W_p^{(m)}(\Omega) \ni u \mapsto \Pi\, u(x) = \int_{K_x} u(y)\, \partial_r^m \psi_x(y) \frac{1}{|x-y|^{n-1}}\, dy \in \mathbb{P}^{m-1}. \quad (1)$$

Here K_x, ψ_x and r are to be explained later (during the talk. See [1]).

Note the norm

$$\|u\|_{W_p^{(m)}} = \left(\sum_{k=0}^{m} |u|_{p,k}^p \right)^{1/p}, \quad u \in W_p^{(m)}(\Omega),$$

with the *homogeneous semi-norm* of order k, $0 \leqq k \leqq m$:

$$|u|_{p,k} = \left(\sum_{|\alpha|=k} \int_\Omega |\partial^\alpha u(x)|^p \, dx \right)^{1/p}.$$

For each homogeneous semi-norm holds the following estimate, which is (actualy very close to one sometimes) called the Bramble-Hilbert Lemma (cf. [3]).

Lemma 1 Suppose Ω is bounded and star-shaped with respect to a closed ball K. Then

$$|u - \Pi u|_{p,k} \leqq C_{n,m,k}(1+\gamma(\Omega))^{n+k} \operatorname{diameter}(\Omega)^{m-k} |u|_{p,m} \tag{2}$$

for $k = 0, \cdots, m-1$. Here $\gamma(\Omega)$ is the *chunkiness parameter* of Ω:

$$\gamma(\Omega) = \frac{\operatorname{diameter}(\Omega)}{\text{maximal radius of closed balls in } \Omega}.$$

Remark 1 (2) is important in discussions of the finite element method. Incidentally, Sobolev's projection operator Π is *not* an orthogonal projection when $p = 2$.

2 A rough explanation of the background

Suppose a bounded domain Ω with a *nice* boundary $\partial\Omega$ is given. It is expected that Ω is appproximated by a sequence of the unions of non-overlapping polygonal subdomains $\Omega_h = \bigcup_k \Delta_{h,k}$, each $\Delta_{h,k}$ being star-shaped with respect to a closed ball inside:

$$\Omega = \lim_h \Omega_h = \lim_h \bigcup_k \overline{\Delta_{k,h}}.$$

A given Sobolev function u on Ω may have polygonal approximation $p_{k,h}$ on each $\Delta_{h,k}$. Errors are estmated using Lemma 1. Then $p_h(x)$ on Ω_h, where

$$p_h(x) = p_{h,k}(x), \quad x \in \Delta_{h,k}$$

may be expected to approximate u within a certain Sobolev space over Ω_h. Then letting Ω_h to Ω, we get a sequence of piecewise polynomial approximations of u. Actually, this depends on Ω and on the choice of the way to define polynomial approximation on each $\Delta_{h,k}$. It is the core of the finite element method ([3]).

On the other hand, this approach may shed some light on a construction of a standard countable dense subset in a Sobolev space, which consists of *computable* elements ([4], [5]). Actually, the conventional finite elements do not appear very convenient, except the simplest Lagrange interpolation (valid in $W_p^{(1)}(\Omega)$ for $u \in W_p^{(2)}(\Omega)$).

That is, given a certain problem of mathematical analysis, formulated in Sobolev spaces, existence and/or uniqueness of the solution is generally obtained by the *classical* method of mathematical analysis together with rather rough behavior of thus obtained solutions. In some case, numerical simulation provides much more detailed knowledge of the solution. There may be some problems for which it is known whether the solutions can basically be *computable* or not, and, in case the solution is computable, to what extent it is computably complex. These questions could not be independent each other, but have hardly been discussed on the *common* ground.

Here is an effort to provide something common.

References

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