Perturbation theorem for $C_0$-groups on Hilbert space

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We consider perturbations of $C_0$-group generators. Let $X$ be a complex Hilbert space. First, we define a $C_0$-group on $X$ and its generator.

**Definition 1.** (i) A one-parameter family $\{T(t); t \in \mathbb{R}\}$ is a $C_0$-group of bounded linear operators if (a) $T(0) = I$; (b) $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}$; (c) $T(t)x \rightarrow x$ (as $t \rightarrow 0$) for all $x \in X$.

(ii) The generator $A$ of a $C_0$-group $\{T(t); t \in \mathbb{R}\}$ is defined as

$$y = Ax \ (x \in D(A)) \iff T(t)x - x = \int_0^t T(s)yds.$$  

An operator $A$ is the generator of a $C_0$-group $\{T(t); t \in \mathbb{R}\}$ satisfying $\|T(t)\| \leq Me^{\gamma|t|}$ for all $t \in \mathbb{R}$ if and only if $A$ is closed, $D(A) = X$ and $\|\lambda \pm A\|^{-1} \leq M(\lambda - \gamma)^{-n}$ for all $\lambda > \gamma \geq 0$ and $n \in \mathbb{N}$.

The next theorem is concerned with perturbations of the generator of a $C_0$-group, however, it involves only first-order resolvent condition.

**Theorem 2.** Let $A$ be the generator of a $C_0$-group on $X$ and $B$ a closed operator on $X$ such that $D(A) \subset D(B)$. Assume that there exist constants $0 < M < 1$ and $\gamma_0 > 0$ such that

(1) $\|B(\gamma + i\omega - A)^{-1}\| \leq M$,
(2) $\|B(\gamma + i\omega - A)^{-1}Bu\| \leq M\|u\|$ for all $\gamma > \gamma_0$ and $\omega \in \mathbb{R}$ (here $i = \sqrt{-1}$). Then $A + B$ generates a $C_0$-group on $X$.

Referring to [1] and [2], we can prove this theorem.

1st step. (1) and (2) imply by [1] that $A + B$ generates a $C_0$-semigroup $\{T(t); t \geq 0\}$ on $X$.

2nd step. (3) guarantees by [2] that $\{T(t)\}$ is embedded into a $C_0$-group on $X$.

The conclusion is thus trivial if we assume further that

(4) $\|B(\gamma + i\omega + A)^{-1}Bu\| \leq M\|u\|$ for all $B \in D(B)$.

Therefore the meaning of Theorem 1 may be stated as follows: only condition (3) is sufficient to extend the semigroup to a group without having recourse to (4).

This theorem is applied to Schrödinger type operators as follows.

**Example.** Let $X := L^2(\mathbb{R})$, $k \in \mathbb{N}$ and consider differential operators of the form:

$$(Au)(x) := i\frac{d^{2k}u}{dx^{2k}}(x), \quad D(A) := H^{2k}(\mathbb{R}),$$

$$(Bv)(x) := V(x)\frac{d^l v}{dx^l}(x), \quad D(B) := H^l(\mathbb{R}),$$

where $V \in H^{l+1}(\mathbb{R})$ and $l = 0, 1, 2, \ldots, k - 1$. Then $A + B$ generates a $C_0$-group on $L^2(\mathbb{R})$.

**References**
