POINTWISE DECAY ESTIMATES OF SOLUTIONS TO THE WAVE EQUATION IN THE EXTERIOR OF OBSTACLES

HIDEO KUBO (OSAKA UNIVERSITY)

This talk is based on the joint work with Prof. S. Katayama (Wakayama University).

Let Ω be an unbounded domain in $\mathbf{R}^3$ with compact and smooth boundary $\partial\Omega$. We put $\mathcal{O} := \mathbf{R}^3 \setminus \Omega$, which is called an obstacle. This talk is concerned with the mixed problem for a system of nonlinear wave equations in Ω, with small initial data:

$$(\partial_t^2 - c_i^2\Delta)u_i = F_i(u, \partial u, \nabla_x \partial u), \qquad (t,x) \in (0,\infty) \times \Omega, \tag{1}$$

$$u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \partial\Omega, \tag{2}$$

$$u(0,x) = \phi(x), \ (\partial_t u)(0,x) = \psi(x), \qquad x \in \Omega, \tag{3}$$

for $i = 1, \dots, N$, where c_i $(1 \le i \le N)$ are given positive constants, and $u = (u_1, \dots, u_N)$. Here we have set $\partial_0 \equiv \partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $(j = 1,2,3)$, $\Delta = \sum_{j=1}^3 \partial_j^2$, $\nabla_x u = (\partial_1 u, \partial_2 u, \partial_3 u)$ and $\partial u = (\partial_t u, \nabla_x u)$. We assume ϕ, $\psi \in C_0^\infty(\overline{\Omega}\,; \mathbf{R}^N)$, namely they are smooth functions on $\overline{\Omega}$ whose support is compact in $\overline{\Omega}$. We suppose (1) is quasi-linear, namely smooth function $F = (F_1, \dots, F_N)$ has the form

$$F_i(u, \partial u, \nabla_x \partial u) = \sum_{j=1}^N c_{ij}^{ka}(u, \partial u)\partial_k\partial_a u_j + \widetilde{F}_i(u, \partial u)$$

with c_{ij}^{ka} and $\widetilde{F}_i$ vanishing of first order and second order at the origin, respectively. In the following we always assume that

$$c_{ij}^{ka}(u, \partial u) = c_{ji}^{ka}(u, \partial u) \text{ and } c_{ij}^{k\ell}(u, \partial u) = c_{ij}^{\ell k}(u, \partial u) \tag{4}$$

hold for $1 \le i,j \le N$, $1 \le k, \ell \le 3$ and $0 \le a \le 3$, so that the hyperbolicity of the system is assured.

We also suppose that (ϕ, ψ, F) satisfies the compatibility condition to infinite order for the mixed problem (1)–(3). Namely, $(\partial_t^j u)(0,x)$, formally determined by (1) and (3), vanishes on $\partial\Omega$ for any non-negative integer j (notice that the values $(\partial_t^j u)(0,x)$ are determined by (ϕ, ψ, F) successively; for example we have $\partial_t^2 u_i(0,x) = c_i^2\Delta\phi_i + F_i\big(\phi, (\psi, \nabla_x\phi), \nabla_x(\psi, \nabla_x\phi)\big)$, and so on).

Let us recall the known results. For $F_i = F_i(u, \partial u, \nabla_x \partial u)$, we denote the quadratic part of F_i by $F_i^{(2)}$. More precisely, writing $Y = (Y_1, \dots, Y_{17N}) = (u, \partial u, \nabla_x \partial u)$, we define

$$F_i^{(2)}(Y) = \sum_{|\alpha|=2} \frac{(\partial_Y^\alpha F_i)(0)}{\alpha!} Y^\alpha. \tag{5}$$

If we suppose that all $F_i^{(2)}$ vanish and that $\mathcal{O}$ is non-trapping, then it was shown in Shibata – Tsutsumi [11] that the mixed problem (1)–(3) with $N = 1$ admits a unique global small amplitude solution. While, if $F_i^{(2)}$ is present, then we need a certain algebraic condition on it in general, in order to get the global existence result. One of such conditions is the null condition introduced by Klainerman [5] when $c_1 = \cdots = c_N$. For the multiple speeds case where the propagation speeds c_i $(1 \le i \le N)$ do not necessarily coincide with each other, we say that the nonlinearity F satisfies the null condition associated with the propagation speeds $(c_1, \dots, c_N)$, if each $F_i^{(2)}$ depends only on ∂u and $\nabla_x \partial u$ (namely $F_i^{(2)} = F_i^{(2)}(\partial u, \nabla_x \partial u)$), and satisfies

$$F_i^{(2)}(V(\mu, X), W(\nu, X)) = 0 \tag{6}$$

for any μ, $\nu \in \Lambda_i$ and $X = (X_0, X_1, X_2, X_3) \in \mathbf{R}^4$ satisfying $X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2)$, where

$$\Lambda_i = \{(\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbf{R}^N; \lambda_j = 0 \text{ if } c_j \ne c_i\}.$$

Here we put $V(\mu, X) = (X_a\, \mu_k : \ a = 0, 1, 2, 3,\ k = 1, \dots, N)$, $W(\nu, X) = (X_j X_a \nu_k : \ j = 1, 2, 3,\ a = 0, 1, 2, 3,\ k = 1, \dots, N)$.

Under the null condition associated with the propagation speeds $(c_1, \dots, c_N)$, the global solvability for the mixed problem with small initial data was shown by Metcalfe – Nakamura – Sogge [8, 9] for rather general obstacle (see also Godin [1], Keel – Smith – Sogge [4], Metcalfe [7], Metcalfe – Sogge [10]).

Our aim is to establish an alternative approach to these works along the same line as [6]. We shall use weighted L^∞-L^∞ estimates for the mixed problem of the linear wave equation which will give us

$$|u_i(t,x)| \le C\varepsilon(1 + t + |x|)^{-1} \log\left(1 + \frac{1 + c_i t + |x|}{1 + |c_i t - |x||}\right), \tag{7}$$

$$|\partial u_i(t,x)| \le C\varepsilon(1 + |x|)^{-1}(1 + |c_i t - |x||)^{-1} \tag{8}$$

for $(t, x) \in [0, \infty) \times \overline{\Omega}$. These estimates are refinement of time decay estimates obtained in the previous works. In this way, we do not need the space–time L^2 estimates which have been adopted in the works [4, 7, 8, 9, 10].

We shall make use of stronger decay property of a tangential derivative to the light cone. This idea enables us to deal with the null form without using neither the scaling operator $t\partial_t + x \cdot \nabla_x$ nor Lorentz boost fields $t\partial_j + x_j\partial_t$ $(j = 1, 2, 3)$. In contrast, the scaling operator has been used in the previous works, and it makes the argument rather complicated because it does not preserve the Dirichlet boundary condition.

We also avoid the argument of a reduction to zero initial data, used in [4, 7, 8, 9, 10].

In order to state our result, we introduce a notion concerning the obstacle. Consider the mixed problem for a single wave equation:

$$
\begin{aligned}
&(9) \qquad (\partial_t^2 - c^2\Delta)v = f, && (t,x) \in (0,T) \times \Omega,\\
&(10) \qquad v(t,x) = 0, && (t,x) \in (0,T) \times \partial\Omega,\\
&(11) \qquad v(0,x) = v_0(x),\ (\partial_t v)(0,x) = v_1(x), && x \in \Omega
\end{aligned}
$$

with some propagation speed $c > 0$. We may assume, without loss of generality, that $\mathcal{O} \subset B_1$ by the scaling. For $R \geq 1$, we set

$$\Omega_R = \Omega \cap \overline{B_R},$$

where $B_R = \{x \in \mathbf{R}^3;\ |x| < R\}$ for $R > 0$.

We denote by $X_c(T)$ the set of all $\Xi = (v_0, v_1, f)$ satisfying $\vec{v}_0 = (v_0, v_1) \in C_0^\infty(\overline{\Omega}; \mathbf{R}^2)$, $f \in C^\infty\big([0,T) \times \overline{\Omega}; \mathbf{R}\big)$, and $\operatorname{supp} f(t,\cdot) \subset \Omega_{ct+R}$ for any $t \in [0,T)$ with some $R > 0$, as well as the compatibility condition to infinite order for (9)–(11). In addition, for $a > 1$, $X_{c,a}(T)$ denotes the set of all $\Xi = (v_0, v_1, f) \in X_c(T)$ satisfying

$$\operatorname{supp} v_0 \cup \operatorname{supp} v_1 \cup \operatorname{supp} f(t,\cdot) \subset \Omega_a \ (t \in [0,T)).$$

Now, we say that the obstacle $\mathcal{O}$ is **admissible** if there exists a non-negative integer ℓ having the following property: Suppose that $\Xi = (\vec{v}_0, f) \in X_{c,a}(T)$ for some $c > 0$ and $a > 1$. Then for any $\gamma > 0$, $b > 1$ and any integer $m \geq 1$, there exists a positive constant $C = C(\gamma, a, b, c, m, \Omega)$ such that for $t \in [0,T)$,

$$
\begin{aligned}
(12) \qquad & \sum_{|\alpha| \leq m} \langle t \rangle^\gamma \, \|\partial^\alpha v(t,\cdot) : L^2(\Omega_b)\| \\
& \leq C\Big(\|\vec{v}_0 : \mathcal{H}^{m+\ell-1}(\Omega)\| + \sup_{0 \leq s \leq t} \langle s \rangle^\gamma \sum_{|\alpha| \leq m+\ell-1} \|\partial^\alpha f(s,\cdot) : L^2(\Omega)\|\Big),
\end{aligned}
$$

where v is the solution to (9)–(11). Here $\langle t \rangle = \sqrt{1+t^2}$ for $t \in \mathbf{R}$, $\partial^\alpha = \partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ for a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and $\mathcal{H}^m(\Omega) = H^{m+1}(\Omega) \times H^m(\Omega)$. We often refer to (12) as local energy decay. We remark that non-trapping obstacles, and trapping obstacles treated in Ikawa [2, 3] are admissible in our sense.

For non-negative integers m and s, we define $H^{m,s}(\Omega) = \{\varphi; \|\varphi : H^{m,s}(\Omega)\| < \infty\}$, where

$$\|\varphi : H^{m,s}(\Omega)\|^2 = \sum_{|\alpha| \le m} \int_\Omega (1+|x|^2)^s |\partial_x^\alpha \varphi(x)|^2 dx$$

for $\varphi = \varphi(x)$.

Our main result reads as follows.

Theorem 1. *Let ϕ, $\psi \in C^\infty(\overline{\Omega}\,; \mathbf{R}^N)$. Suppose that $\mathcal{O}$ is admissible, that (ϕ, ψ, F) satisfies the compatibility condition to infinite order for the problem* (1)–(3)*, and that F satisfies the null condition associated with $(c_1, \ldots, c_N)$. Then there exist a positive constant ε_0 and a large integer s such that the mixed problem* (1)–(3) *admits a unique solution $u \in C^\infty([0,\infty) \times \overline{\Omega}; \mathbf{R}^N)$ satisfying* (7) *and* (8)*, provided that $\|\phi : H^{s+2,s}(\Omega)\| + \|\psi : H^{s+1,s}(\Omega)\| \le \varepsilon_0$.*

References

[1] P. Godin, *Global existence of solutions to some exterior radial quasilinear Cauchy-Dirichlet problems*, Amer. J. Math. **117** (1995), 1475–1505.

[2] M. Ikawa, *Decay of solutions for the wave equation in the exterior of two convex bodies*, Osaka J. Math. **19** (1982), 459–509.

[3] M. Ikawa, *Decay of solutions for the wave equation in the exterior of several convex bodies*, Ann. Inst. Fourier (Grenoble), **38** (1988), 113–146.

[4] M. Keel, H. Smith and C. D. Sogge, *Global existence for a quasilinear wave equation outside of star–shaped domains*, J. Funct. Anal. **189** (2002), 155–226.

[5] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lectures in Applied Math.**23** (1986), 293–326.

[6] H. Kubo, *Uniform decay estimates for the wave equation in an exterior domain*, To appear in Advanced Studies in Pure Mathematics, Math. Soc. of Japan.

[7] J. Metcalfe, *Global existence for semilinear wave equations exterior to non-trapping obstacles*, Houston J. Math. **30** (2004), 259–281.

[8] J. Metcalfe, M. Nakamura and C. D. Sogge, *Global existence of solutions to multiple speed systems of quasilinear wave equations in exterior domains*, Forum Math. **17** (2005), 133–168.

[9] J. Metcalfe, M. Nakamura and C. D. Sogge, *Global existence of quasilinear, nonrelativistic wave equations satisfying the null condition*, Japan. J. Math. (N.S.) **31** (2005), 391–472.

[10] J. Metcalfe and C. D. Sogge, *Hyperbolic trapped rays and global existence of quasilinear wave equations*, Invent. Math.

[11] Y. Shibata and Y. Tsutsumi, *On a global existence theorem of small amplitude solutions for nonlinear wave equations in an exterior domain*, Math. Z **191** (1986), 165–199.