On Blow-up at Space Infinity for Nonlinear Heat Equations

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We are interested in solutions of nonlinear heat equations which blow up at space infinity. We consider solutions of the initial value problem for the equation

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbf{R}^n, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^n. \end{cases} \tag{1}$$

The nonlinear term f is assumed to be locally Lipschitz in $[0, \infty)$ and a nonnegative function satisfying

$$f(\delta b) \leq \delta^p f(b). \tag{2}$$

for all $b \geq b_0$ and for all $\delta \in (\delta_0, 1)$ with some $b_0 > 0$, some $\delta_0 \in (0,1)$ and some $p > 1$. The initial data u_0 is assumed to be a measureable function in $\mathbf{R}^n$ satisfying

$$0 \leq u_0 \leq M \text{ a.e.} \quad \text{and} \quad u_0 \not\equiv M \text{ a.e.} \tag{3}$$

with some positive M. We are interested in initial data such that $u_0 \to M$ as $|x| \to \infty$ for x in some sector of $\mathbf{R}^n$. We assume that

$$\lim_{m \to \infty} u_0(x + x_m) = M \quad \text{a.e. in } \mathbf{R}^n. \tag{4}$$

with sequence $\{x_m\}_{m=1}^{\infty}$ is some sequence of vector. (In fact, it follows from (3) that $|x_m| \to \infty$ as $m \to \infty$.)

Problem (1) has a unique bounded solution at least locally in time. However, the solution may blow up in finite time. For a given initial value u_0 and nonlinear term f let $T^* = T^*(u_0, f)$ be the maximal existence time of the solution. If $T^* = \infty$, the solution exists globally in time. If $T^* < \infty$, we say that the solution blows up in finite time. It is well known that

$$\limsup_{t \to T^*} \|u(\cdot, t)\|_\infty = \infty. \tag{5}$$

In this paper, we are interested in behavior of a blowing up solution near space infinity as well as location of blow-up points defined below. A point $x_{BU} \in \mathbf{R}^n$ is called a *blow-up point* (with value $\pm\infty$) if there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad x_m \to x_{BU} \quad \text{and} \quad u(x_m, t_m) \to \pm\infty \quad \text{as} \quad m \to \infty.$$

If there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ such that

$$t_m \uparrow T^*, \quad |x_m| \to \infty \quad \text{and} \quad u(x_m, t_m) \to \pm\infty \quad \text{as} \quad m \to \infty,$$

then we we say that the solution blows up to $\pm\infty$ at space infinity.

A direction $\psi \in S^{n-1}$ is called a *blow-up direction for the value* $\pm\infty$ if there exists a sequence $\{(x_m, t_m)\}_{m=1}^{\infty}$ with $x_m \in \mathbf{R}^n$ and $t_m \in (0, T^*)$ such that $u(x_m, t_m) \to \pm\infty$ (as $m \to \infty$) and

$$\frac{x_m}{|x_m|} \to \psi \quad \text{as} \quad m \to \infty. \tag{6}$$

We consider the solution $v(t)$ of an ordinary differential equation

$$\begin{cases} v_t = f(v), & t > 0, \\ v(0) = M. \end{cases} \tag{7}$$

Let $T_v = T^*(M, f)$ be the maximal existence time of solutions of (7), i. e.,

$$T_v = \int_M^\infty \frac{ds}{f(s)}.$$

We are now in position to state our main results.

Theorem 1. *Assume that f is locally Lipschitz in $\boldsymbol{R}$ and satisfies* (2)*. Let u_0 be a continuous function satisfying* (3) *and* (4)*. Then there exists a subsequence of $\{x_m\}_{m=1}^{\infty}$ (still denote by $\{x_m\}$, independent of t) such that*

$$\lim_{m \to \infty} u(x + x_m, t) = v(t) \quad \text{a.e. in } \mathbf{R}^n.$$

The convergence is uniform in every compact subset of $\{t : 0 \le t < T_v\}$. Moreover, the solution blows up at T_v.

Theorem 2. *Assume the same hypotheses of Theorem 1. Then the solution of* (1) *has no blow-up points in* $\boldsymbol{R}^n$. (*It blows up only at space infinity.*)

Theorem 3. *Assume the same hypotheses of Theorem 1. Let a direction* $\psi \in S^{n-1}$. *If and only if there exists a sequence* $\{y_m\}_{m=1}^{\infty}$ *satisfying* $\lim_{m\to\infty} y_m/|y_m| = \psi$ *such that*

$$u_0(x + y_m) \to M \text{ as } m \to \infty \text{ a.e. in } \mathbf{R}^n,$$

then ψ *is a blow-up direction.*

References

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