Doubly nonlinear evolution equations and dynamical systems

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1 Introduction

Let V and V^* be a reflexive Banach space and its dual space, respectively, and let H be a Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^* \tag{1}$$

with continuous and densely defined canonical injections. Let φ and ψ be proper lower semi-continuous functions from V into $(-\infty, \infty]$, and let $\partial_V \varphi, \partial_V \psi : V \to 2^{V^*}$ be subdifferential operators of φ and ψ respectively. Moreover, let B be a (possibly) non-monotone and multi-valued operator from V into V^*.

This talk deals with the dynamical system generated by the Cauchy problem (CP) for the following doubly nonlinear evolution equation:

$$\partial_V \psi(u'(t)) + \partial_V \varphi(u(t)) + B(u(t)) \ni f \quad \text{in } V^*, \quad 0 < t < \infty, \tag{2}$$

where $f \in V^*$ and $u_0 \in D(\varphi) := \{u \in V; \varphi(u) < \infty\}$ are given data. We first prove the existence of global (in time) strong solutions of (CP) by imposing appropriate conditions such as the coerciveness and the boundedness of $\partial_V \psi$, the precompactness of sub-level sets of φ, and the boundedness and the compactness of B. The main purpose of this talk is to discuss the large-time behavior of global solutions for (CP), in particular, the existence of global attractors; however, since the scope of our abstract framework involves the case where (CP) admits multiple solutions, the usual semi-group approach to dynamical systems could be no longer valid. Therefore we employ the notion of generalized semiflow proposed by J.M. Ball [3] to treat global attractors for (CP).

Furthermore, we apply the preceding abstract theory to generalized Allen-Cahn equations. Gurtin [4] proposed a generalized Allen-Cahn equation, which describes the evolution of an order parameter $u = u(x, t)$, of the form

$$\rho(u, \nabla u, u_t) u_t = \operatorname{div} \left[\partial_{\mathbf{p}} \hat{\psi}(u, \nabla u) \right] - \partial_r \hat{\psi}(u, \nabla u) + f, \tag{3}$$

where $\rho = \rho(r, \mathbf{p}, s) \geq 0$ is a constitutive modulus, $\hat{\psi} = \hat{\psi}(r, \mathbf{p})$ denotes a free energy density and f is an external microforce. As a simple instance of the free energy density $\hat{\psi}$, we often take

$$\hat{\psi}(r, \mathbf{p}) = \frac{1}{2} |\mathbf{p}|^2 + W(r)$$

with a double-well potential $W(r) = (r^2 - 1)^2$. In this talk we treat a generalized Allen-Cahn equation of degenerate type as well as a perturbation problem of a semilinear generalized Allen-Cahn equation.

2 Generalized semiflow

The notion of generalized semiflow was first introduced by J.M.Ball [3]. He also defined global attractors for generalized semiflows and provided a criterion of the existence of global attractors. We first recall the definition of generalized semiflow.

Definition 2.1. *Let X be a metric space with metric $d_X = d_X(\cdot,\cdot)$. A family $\mathcal{G}$ of maps $\varphi : [0,\infty) \to X$ is said to be a generalized semiflow in X, if the following four conditions are all satisfied:*

(H1) (Existence) *for each $x \in X$ there exists $\varphi \in \mathcal{G}$ such that $\varphi(0) = x$;*

(H2) (Translation invariance) *if $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then the map φ^τ also belongs to $\mathcal{G}$, where $\varphi^\tau(t) := \varphi(t+\tau)$ for $t \in [0,\infty)$;*

(H3) (Concatenation invariance) *if $\varphi_1, \varphi_2 \in \mathcal{G}$ and $\varphi_2(0) = \varphi_1(\tau)$ at some $\tau \geq 0$, then the map ψ, the concatenation of φ_1 and φ_2 at τ, defined by*

$$\psi(t) := \begin{cases} \varphi_1(t) & \text{if } t \in [0,\tau], \\ \varphi_2(t-\tau) & \text{if } t \in (\tau,\infty) \end{cases}$$

also belongs to $\mathcal{G}$;

(H4) (Upper semicontinuity) *if $\varphi_n \in \mathcal{G}$, $x \in X$ and $\varphi_n(0) \to x$ in X, then there exist a subsequence (n') of (n) and $\varphi \in \mathcal{G}$ such that $\varphi_{n'}(t) \to \varphi(t)$ for each $t \in [0,\infty)$.*

Let $\mathcal{G}$ be a generalized semiflow in a metric space X. We define a map $T(t) : 2^X \to 2^X$ by

$$T(t)E := \{\varphi(t);\ \varphi \in \mathcal{G} \text{ and } \varphi(0) \in E\} \quad \text{for } E \subset X \tag{4}$$

for each $t \geq 0$. Moreover, global attractors for generalized semiflows are defined as follows.

Definition 2.2. *Let $\mathcal{G}$ be a generalized semiflow in a metric space X and let $(T(t))_{t\geq 0}$ be the family of mappings defined as in (4). A set $\mathcal{A} \subset X$ is said to be a global attractor for the generalized semiflow $\mathcal{G}$ if the following* (i)–(iii) *hold.*

(i) *$\mathcal{A}$ is compact in X;*

(ii) *$\mathcal{A}$ is invariant under $T(t)$, i.e., $T(t)\mathcal{A} = \mathcal{A}$, for all $t \geq 0$;*

(iii) *$\mathcal{A}$ attracts any bounded subsets B of X by $(T(t))_{t\geq 0}$, i.e.,*

$$\lim_{t\to\infty} \operatorname{dist}(T(t)B, \mathcal{A}) = 0,$$

where $\operatorname{dist}(\cdot,\cdot)$ *is defined by*

$$\operatorname{dist}(A,B) := \sup_{a\in A} \inf_{b\in B} d_X(a,b) \quad \text{for } A, B \subset X.$$

3 Main results

Let us first state our basic assumptions: let $p \in (1,\infty)$, $T > 0$ and $\varepsilon > 0$ be fixed.

(A1) There exist positive constants C_i $(i = 1,2,3,4)$ such that

$$C_1|u|_V^p \leq \psi(u) + C_2 \quad \text{for all } u \in D(\psi),$$
$$|\eta|_{V^*}^{p'} \leq C_3\psi(u) + C_4 \quad \text{for all } [u,\eta] \in \partial_V\psi.$$

(A2) There exist a reflexive Banach space X_0 and a non-decreasing function ℓ_1 on $[0,\infty)$ such that X_0 is compactly embedded in V and

$$|u|_{X_0} \leq \ell_1(|u|_H + [\varphi(u)]_+) \quad \text{for all } u \in D(\partial_V\varphi),$$

where $[s]_+ := \max\{s,0\} \geq 0$ for $s \in \mathbb{R}$.

(A3)$_\varepsilon$ $D(\partial_V\varphi) \subset D(B)$. There exists a constant $c_\varepsilon \geq 0$ such that

$$|g|_{V^*}^{p'} \leq \varepsilon|\xi|_{V^*}^{\sigma} + c_\varepsilon\left\{|\varphi(u)| + |u|_V^p + 1\right\} \quad \text{with } \sigma := \min\{2, p'\}$$

for all $u \in D(\partial_V\varphi)$, $g \in B(u)$ and $\xi \in \partial_V\varphi(u)$.

(A4) Let $S \in (0, T]$ and let (u_n) and (ξ_n) be sequences in $C([0, S]; V)$ and $L^\sigma(0, S; V^*)$ with $\sigma := \min\{2, p'\}$, respectively, such that $u_n \to u$ strongly in $C([0, S]; V)$, $[u_n(t), \xi_n(t)] \in \partial_V\varphi$ for a.e. $t \in (0, S)$, and

$$\sup_{t\in[0,S]} |\varphi(u_n(t))| + \int_0^S |u_n'(t)|_H^p dt + \int_0^S |\xi_n(t)|_{V^*}^{\sigma} dt \quad \text{is bounded for all } n \in \mathbb{N},$$

and let (g_n) be a sequence in $L^{p'}(0, S; V^*)$ such that $g_n(t) \in B(u_n(t))$ for a.e. $t \in (0, S)$ and $g_n \to g$ weakly in $L^{p'}(0, S; V^*)$. Then (g_n) is precompact in $L^{p'}(0, S; V^*)$ and $g(t) \in B(u(t))$ for a.e. $t \in (0, S)$.

(A5) Let $S \in (0, T]$ and $u \in C([0, S]; V) \cap W^{1,p}(0, S; H)$ be such that $\sup_{t\in[0,S]} |\varphi(u(t))| < \infty$ and suppose that there exists $\xi \in L^{p'}(0, S; V^*)$ such that $\xi(t) \in \partial_V\varphi(u(t))$ for a.e. $t \in (0, S)$. Then there exists a V^*-valued strongly measurable function g such that $g(t) \in B(u(t))$ for a.e. $t \in (0, S)$. Moreover, the set $B(u)$ is convex for all $u \in D(B)$.

Here we are concerned with the strong solutions of (CP) given as follows:

Definition 3.1. *For $T \in (0, \infty)$, a function $u \in AC([0, T]; V)$ is said to be a strong solution of* (CP) *on $[0, T]$, if the following conditions are satisfied*:

(i) $u(0) = u_0$,

(ii) *there exists a negligible set $N \subset (0, T)$, i.e., the Lebesgue measure of N is zero, such that $u(t) \in D(\partial_V\varphi)$ and $u'(t) \in D(\partial_V\psi)$ for all $t \in [0, T] \setminus N$, and moreover, there exist sections $\eta(t) \in \partial_V\psi(u'(t))$, $\xi(t) \in \partial_V\varphi(u(t))$ and $g(t) \in B(u(t))$ such that*

$$\eta(t) + \xi(t) + \lambda g(t) = f \quad \text{in } V^* \text{ for all } t \in [0, T] \setminus N, \tag{5}$$

(iii) $u(t) \in D(\varphi)$ *for all $t \in [0, T]$, and the function $\varphi(u(\cdot))$ is absolutely continuous on $[0, T]$.*

Furthermore, for $T \in (0, \infty]$, a function $u \in AC([0, T); V)$ is said to be a strong solution of (CP) *on $[0, T)$, if u is a strong solution of* (CP) *on $[0, S]$ for every $S \in (0, T)$.*

The following theorem is concerned with the existence of global (in time) strong solutions.

Theorem 3.2 (Global existence)**.** *Let $p \in (1, \infty)$ and $T > 0$ be fixed. Suppose that* (A1)–(A5) *are all satisfied with a sufficiently small $\varepsilon > 0$. Then, for all $f \in V^*$ and $u_0 \in D(\varphi)$, there exists at least one strong solution $u \in W^{1,p}(0, T; V)$ on $[0, T]$.*

We set $X := D(\varphi)$ with the distance $d_X(u, v) := |u - v|_V + |\varphi(u) - \varphi(v)|$ for $u, v \in X$, and moreover, we define

$$\mathcal{G} := \{u \in C([0, \infty); V);\ u \text{ is a strong solution of (2) on } [0, \infty)\}.$$

Theorem 3.3. *Let $p \in (1, \infty)$ be given. Suppose that* (A1)–(A5) *are all satisfied with a sufficient small constant $\varepsilon > 0$ for any $T > 0$. Then, for all $f \in V^*$, the set $\mathcal{G}$ is a generalized semiflow on X.*

Now, our main result reads,

Theorem 3.4. *Suppose that*

(A6) *There exist constants* $\alpha > 0$ *and* $C_5 \geq 0$ *such that*

$$\alpha \left\{ \varphi(u) + |u|_V^p \right\} \leq \langle \xi + g, u \rangle + C_5$$

for all $u \in D(\partial_V \varphi)$, $\xi \in \partial_V \varphi(u)$ *and* $g \in B(u)$.

In addition, assume $f \in V^*$ *and* (A1)–(A5) *with an enough small constant* $\varepsilon > 0$ *for any* $T > 0$. *Then the generalized semiflow* $\mathcal{G}$ *has a global attractor* $\mathcal{A}$, *and* $\mathcal{A}$ *is a unique maximal compact invariant subset of* X.

4 Applications to generalized Allen-Cahn equations

Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$. For given functions $u_0, f : \Omega \to \mathbb{R}$, we first deal with

$$\left.\begin{array}{ll} \alpha(u_t(x,t)) - \Delta_m u(x,t) + \partial_r W(x, u(x,t)) \ni f(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,t) = u_0(x), & x \in \Omega, \end{array}\right\} \qquad (6)$$

where α is a maximal monotone graph in $\mathbb{R}^2$ satisfying a p-power growth condition (e.g., $\alpha(r) = |r|^{p-2}r$) with $p \geq 2$ and Δ_m stands for the so-called m-Laplace operator given by

$$\Delta_m u(x) = \nabla \cdot \left(|\nabla u(x)|^{m-2} \nabla u(x) \right), \quad 1 < m < \infty.$$

Moreover, $\partial_r W$ stands for the derivative in r of a potential $W = W(x,r) : \Omega \times \mathbb{R} \to (-\infty, +\infty]$ given by

$$W(x,r) := j(r) + \int_0^r g(x,\rho) d\rho \quad \text{for } \ x \in \Omega, \ r \in \mathbb{R} \qquad (7)$$

with a lower semicontinuous convex function $j : \mathbb{R} \to (-\infty, +\infty]$ and a (possibly non-monotone) function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ integrable in $\mathbb{R}$. Hence $\partial_r W(x,r) = \partial j(r) + g(x,r)$. Then (6) is regarded as a special case of (3); more precisely, $\rho = \rho(s)$ and $\alpha(s) := \rho(s)s$ is maximal monotone and satisfies a p-power growth condition, and furthermore,

$$\hat{\psi}(r, \mathbf{p}) = \frac{1}{m} |\mathbf{p}|^m + W(r).$$

Let us introduce the following assumptions.

(a1) $g = g(x,r)$ is a Carathéodory function, i.e., measurable in x and continuous in r. Moreover, there exist constants $q \geq 2$, $C_6 \geq 0$ and a function $a_1 \in L^1(\Omega)$ such that

$$|g(x,r)|^{p'} \leq C_6 |r|^{p'(q-1)} + a_1(x)$$

for a.e. $x \in \Omega$ and all $r \in \mathbb{R}$.

(a2) there exist constants $\sigma > 1$ and $C_7 \geq 0$ such that

$$|r|^\sigma \leq C_7 \Big(j(r) + 1 \Big) \quad \text{for all } \ r \in \mathbb{R}.$$

Then our result reads,

Theorem 4.1. *In addition to* (a1) *and* (a2), *assume that*

$$2 \le p < \max\{m^*, \sigma\} \quad \text{and} \quad p'(q-1) < \max\{m, p, \sigma\},$$

where m^* *is the Sobolev critical exponent, i.e.,* $m^* := Nm/(N-m)_+$*. Then, for* $f \in L^{p'}(\Omega)$ *and* $u_0 \in X := \{v \in W_0^{1,m}(\Omega);\ j(v(\cdot)) \in L^1(\Omega)\}$, *the initial-boundary value problem* (6) *admits at least one* L^p*-solution on* $[0, \infty)$. *Moreover, the set of solutions for* (6) *forms a generalized semiflow* $\mathcal{G}$ *on* X. *Furthermore, if* $p \le \max\{m, \sigma\}$, *then* $\mathcal{G}$ *possesses a global attractor on* X.

We next consider the following generalized problem.

$$\left.\begin{array}{ll} \alpha(u_t(x,t)) - \Delta u(x,t) + N(x, u(x,t), \nabla u(x,t)) \ni f(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,t) = u_0(x), & x \in \Omega, \end{array}\right\} \tag{8}$$

where $N = N(x, r, \mathbf{p})$ is written as follows

$$N(x, r, \mathbf{p}) = \partial j(r) + h(x, r, \mathbf{p}) \quad \text{for } x \in \Omega,\ r \in \mathbb{R},\ \mathbf{p} \in \mathbb{R}^N.$$

It could be emphasized that this problem may not be written as a (generalized) gradient system such as (3), since the nonlinear term N depends on the gradient of u. We discuss the existence of global (in time) solutions and their long-time behavior for (6) and (8).

Now we introduce

(a2)′ ∂j is single-valued, and (a2) holds.

(a3) $h = h(x, r, \mathbf{p})$ is a Carathéodory function, i.e., measurable in x and continuous in r and $\mathbf{p}$. There exist constants $q_1, q_2 \ge 2$, $C_3 \ge 0$ and a function $a_2 \in L^1(\Omega)$ such that

$$|h(x, r, \mathbf{p})|^{p'} \le C_3 \left(|r|^{p'(q_1-1)} + |\mathbf{p}|^{p'(q_2-1)} \right) + a_2(x)$$

for a.e. $x \in \Omega$ and all $r \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^N$.

Then we can assure

Theorem 4.2. *In addition to* (a2)′ *and* (a3), *assume that*

$$2 \le p < \max\{2^*, \sigma\}, \quad p'(q_1 - 1) < \max\{p, \sigma\} \quad \text{and} \quad p'(q_2 - 1) < 2.$$

Then, for $f \in L^{p'}(\Omega)$ *and* $u_0 \in X := \{v \in H_0^1(\Omega);\ j(v(\cdot)) \in L^1(\Omega)\}$, *the initial-boundary value problem* (8) *admits at least one* L^p*-solution on* $[0, \infty)$. *Moreover, the set of solutions for* (8) *forms a generalized semiflow* $\mathcal{G}$ *on* X. *Furthermore, if* $p = 2$ *or* $p \le \sigma$, *then* $\mathcal{G}$ *possesses a global attractor on* X.

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