Rotation number approach to spectral analysis of the generalized Kronig-Penney Hamiltonians

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In this talk we study the spectrum of the one-dimensional Schrödinger operators with periodic singular potentials. We fix $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let $0 = \kappa_0 < \kappa_1 < \cdots < \kappa_n = 2\pi$ be a partition of the interval $(0, 2\pi)$. We put $\Gamma_j = \{\kappa_j\} + 2\pi\mathbb{Z}$ for $j = 1, 2, \dots, n$ and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$. For $\{A_j\}_{j=1}^n \subset SL(2, \mathbb{R})$, we define the one-dimensional Schrödinger operator $H = H(A_1, A_2, \dots, A_n)$ in $L^2(\mathbb{R})$ as follows.

$$(Hy)(x) = -\frac{d^2}{dx^2} y(x), \qquad x \in \mathbb{R} \setminus \Gamma,$$

$$\mathrm{Dom}(H) = \left\{ y \in H^2(\mathbb{R} \setminus \Gamma) \;\middle|\; \begin{array}{c} \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = A_j \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \\ \text{for} \quad x \in \Gamma_j, \quad j = 1, 2, \dots, n \end{array} \right\}.$$

The operator H is self-adjoint and is called the generalized Kronig-Penney Hamiltonians. We label each band according to the Floquet-Bloch theory. For $j \in \mathbb{N}$, we designate the jth band of $\sigma(H)$ as $B_j = [\lambda_{2j-2}, \lambda_{2j-1}]$. We have

$$\sigma(H) = \bigcup_{j=1}^{\infty} B_j.$$

The consequtive bands B_j and B_{j+1} are separated by an open interval $G_j = (\lambda_{2j-1}, \lambda_{2j})$, which is called the jth gap of $\sigma(H)$.

The rotation number has a close relation to the spectrum of H. In order to introduce the rotation number, we consider the Schrödinger equation

$$-\frac{d^2}{dx^2} y(x, \lambda) = \lambda y(x, \lambda), \qquad x \in \mathbb{R} \setminus \Gamma, \tag{1}$$

$$\begin{pmatrix} y(x+0, \lambda) \\ y'(x+0, \lambda) \end{pmatrix} = A_j \begin{pmatrix} y(x-0, \lambda) \\ y'(x-0, \lambda) \end{pmatrix}, \qquad x \in \Gamma_j, \quad j = 1, 2, \dots, n, \tag{2}$$

where λ is a real parameter. We define the Prüfer transform of a nontrivial solution $y(x, \lambda)$ to (1) and (2) Let (r, ω) be the polar coordinates of (y, y'):

$$y = r \sin \omega, \quad y' = r \cos \omega.$$

Then we call the function $\omega = \omega(x, \lambda)$ the Prüfer transform of $y(x, \lambda)$. The function $\omega(x, \lambda)$ satisfies the equation

$$\omega'(x, \lambda) = \cos^2 \omega(x, \lambda) + \lambda \sin^2 \omega(x, \lambda), \quad x \in \mathbb{R} \setminus \Gamma \tag{3}$$

as well as the boundary conditions

$$\begin{aligned} &\sin \omega(x + 0, \lambda)(c_j \sin \omega(x - 0, \lambda) + d_j \cos \omega(x - 0, \lambda)) \\ &\quad = \cos \omega(x + 0, \lambda)(a_j \sin \omega(x - 0, \lambda) + b_j \cos \omega(x - 0, \lambda)), \end{aligned} \tag{4}$$

$$\operatorname{sgn}(\sin \omega(x + 0, \lambda)) = \operatorname{sgn}(a_j \sin \omega(x - 0, \lambda) + b_j \cos \omega(x - 0, \lambda)), \tag{5}$$

$$\operatorname{sgn}(\cos \omega(x + 0, \lambda)) = \operatorname{sgn}(c_j \sin \omega(x - 0, \lambda) + d_j \cos \omega(x - 0, \lambda)) \tag{6}$$

for $x \in \Gamma_j$ and $j = 1, 2, \ldots, n$. Let $\omega(x, \lambda, \omega_0)$ be the solution of (3) – (6) subject to the initial condition $\omega(+0, \lambda) = \omega_0 \in \mathbb{R}$. We choose the branch of $\omega(x + 0, \lambda, \omega_0)$ as

$$-\pi \leq \omega(x + 0, \lambda, \omega_0) - \omega(x - 0, \lambda, \omega_0) < \pi \quad \text{for} \quad x \in \Gamma.$$

We define the rotation number of (1) and (2) as

$$\rho(\lambda) = \lim_{n \to \infty} \frac{\omega(2n\pi + 0, \lambda, \omega_0) - \omega_0}{2n\pi}.$$

For $j \in \{1, 2, \cdots, n\}$, we put

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix},$$

and

$$l = \sharp\{1 \leq j \leq n| \quad (b_j < 0) \quad \text{or} \quad (b_j = 0, \quad d_j < 0)\},$$

We have the following results in [4].

THEOREM **1.** *For $j \in \mathbb{N}$, we have*

$$\begin{aligned} \lambda_{2j-2} &= \max\left\{\lambda \in \mathbb{R} \middle| \quad \rho(\lambda) = \frac{j-1}{2} - \frac{l}{2}\right\}, \\ \lambda_{2j-1} &= \min\left\{\lambda \in \mathbb{R} \middle| \quad \rho(\lambda) = \frac{j}{2} - \frac{l}{2}\right\}. \end{aligned}$$

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