On the ground state of nonlinear Schödinger equation with potential

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In this talk, we consider the symmetry breaking phenomena of the ground state of the following nonlinear Schrödinger equation.

$$i\partial_t u = -\Delta u + Vu - |u|^{p-1}u \quad (t,x) \in \mathbb{R}^{1+N}, \tag{1}$$

where $p \in (1, 1+4/N)$, $V \in C^2(\mathbb{R}^N; \mathbb{R})$.

Nonlinear Schödinger equations appear in various region of mathematical physics such as nonlinear optics and Bose-Einstein condensation (BEC).

We are interested in a special solutions of (1) which are expressed as $u(t,x) = e^{i\omega t}\phi_\omega(x)$. These solutions are called standing waves and ω is called the frequency. If $e^{i\omega t}\phi_\omega$ is a standing wave, ϕ_ω satisfies the following stationary problem.

$$0 = -\Delta\phi + (V+\omega)\phi - |\phi|^{p-1}\phi. \tag{2}$$

Assuming appropriate conditions on V, (1) conserves the energy and L^2-norm, where the energy is the following functional.

$$\mathcal{E}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} \, dx.$$

We call the minimizer of the energy in the constraint $\|u\|_{L^2} = \alpha$, the ground state.

Definition 1 (Ground states). *Let*

$$H_V^1 := \left\{ u \in H^1(\mathbb{R}^N) \mid V|u|^2 \in L^1(\mathbb{R}^N) \right\}.$$

We denote the set of the ground state with $\|u\|_{L^2} = \alpha$, $\mathcal{G}_\alpha$.

$$\mathcal{G}_\alpha := \left\{ u \in H_V^1 \mid \|u\|_{L^2} = \alpha, \ \mathcal{E}(u) = \inf_{\|v\|_{L^2}=\alpha} \mathcal{E}(v) \right\}$$

Remark 1. By the Lagrange multiplier method, the ground state is a standing wave with frequency ν_α, which is the Lagrange multiplier. The existence of the ground states are well known [2].

We consider the relation between the symmetry of V and the symmetry of the ground state. For example, if V is radially symmetric, then are the ground states also radially symmetric? For the nonlinear Hartree equation, Aschbacher

et al. [1] showed that for large L^2-norm α, the symmetry some times breaks because of the strong attractive nonlinearity. By the attractive nonlinearity, the ground states concentrates its L^2-norm to the one of the minimum of V. So, in the case V is radially symmetric and does not attain its minimum at the origin, then, by the concentration, the ground state cannot be radially symmetric.

We investigate this concentration result more precisely. Our results are the following.

Theorem 1. *Assume that V satisfies the following conditions.*

(V1) $0 = \inf_{x \in \mathbb{R}^N} V(x) < \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) \leq \infty$.

(V2) $|\nabla V(x)| \leq c_1 + c_2 V(x)$.

(i) *For sufficiently large $\alpha > 0$, every ground state $u \in \mathcal{G}_\alpha$ has only one local maximum $y_{\alpha,u}$ (which is a global maximum). Further, we have*

$$|u(x)| \leq C_1 \alpha^{\frac{4}{4-N(p-1)}} \exp\left(-C_2 \alpha^{\frac{2(p-1)}{4-N(p-1)}} |x - y_{\alpha,u}|\right),$$

where C_1 and C_2 are positive constants independent of α.

(ii) *Further, assume*

(V3) *For every multi-index $|\alpha| \leq 2$, $|\partial^\alpha V(x)| \leq C(1+|x|)^r$ for some $r > 0$.*

Then, as $\alpha \to \infty$ the unique local maximum $y_{\alpha,u}$ converges to Θ_V, where

$$\Theta_V := \left\{ x \in V^{-1}(\{0\}) \mid \Delta V(x) = \min_{y \in V^{-1}(\{0\})} \Delta V(y) \right\}.$$

Corollary 1. *Assume (V1)-(V3). Suppose V is radially symmetric and $0 \notin \Theta_V$. Then, for sufficiently large α, the ground states $u \in \mathcal{G}_\alpha$ are not radially symmetric.*

References

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[2] Thierry Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003. MR MR2002047 (2004j:35266)