On spherically symmetric Einstein equations and its generalization

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Introduction The Einstein equations which is the fundamental ones for general relativity, have solutions with spacetime singularities in physical models. A simple question is how generic there are such singularities. For this, Penrose has shown the following *singularity theorem*:

Theorem 1 (Penrose [PR65]) *If in the initial data set $\{\Sigma, h, k\}$, Σ is non-compact and contains a closed trapped surface $\mathcal{S}$, then the corresponding maxmal future development is incomplete.*

Here, A *closed trapped surface* is a compact spacelike $(n-2)$-dimensional surface such that a displacement (area element) of $\mathcal{S}$ in M along the congruence of the future outgoing null directions decreases. This theorem says that gravitational collapse causes formation of spacetime singularities, but does not say that black holes which are complement regions of causal past of complete future null infinity, should be formed. Then, *predictability* would be breakdown if singularity can be seen from observers at such infinity. For this, the following *weak cosmic censorship (WCC) conjecture* is proposed:

Conjecture 1 (Penrose [PR69], Christodoulou [CD], Klainerman [KS]) *For generic asymptotically flat Cauchy data, solutions to the Einstein-matter equations possess a complete null infinity.*

Remark 1 *This formulation is of Christodoulou. The original is formulated by Penrose.*

Remark 2 *We can find more informations about spacetime singularities, cosmic censorship and the initial value problem of the Einstein equations in the recent text book by Rendall [RA].*

To prove the WCC, one need to show (1) global existence theorems in suitable coordinates and (2) completeness of null infinity (analyzing asymptotic behavior of the solutions). However, it is too difficult to solve the Einstein-matter equations without assumptions. Therefore, *spherical symmetry* is assumed as a typical example. By Birkhoff's theorem which states that spherically symmetric vacuum solutions to the Einstein equations should be static, matter fields are needed to generate dynamics. The most simplest matter model is *massless* *scalar field*. In this case, Christodoulou has proved the WCC [CD]. Dafermos has generalized this to the case of *nonlinear scalar fields* [DMa, DMb]. Moreover, the case of *wave maps* has been considered [Na]. We would like to extend these result to more general gravitational theory which arising in the unified theories such as superstring [Nb].

Einstein-Gauss-Bonnet equations Let $(M, g_{\mu\nu})$ be a spacetime, where M is an orientable n-dimensional smooth manifold and $g_{\mu\nu}$ a Lorentzian metric on it[1] . The action we will consider is

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{2\kappa_n^2} \left(-R - \alpha L_2 \right) + \mathcal{L}_m(\Psi^A, \partial \Psi^A) \right], \tag{1}$$

where κ_n^2 is the Newton constant, R is the Ricci scalar and $\mathcal{L}_m$ is the Lagrangian density of matter fields Ψ^A. The *Gauss-Bonnet* term L_2 is given as

$$L_2 := R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma},$$

where α is a nonnegative coupling constant, $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the Ricci and Riemann tensors, respectively. Varying this action with respect to the metric and matter fields, we have the Einstein-Gauss-Bonnet(EGB)-matter equations as follow:

$$G_{\mu\nu} + \alpha H_{\mu\nu} = \kappa_n^2 T_{\mu\nu}, \tag{2}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}_m}{\partial(\partial_\mu \Psi^A)} \right) - \frac{\partial \mathcal{L}_m}{\partial \Psi^A} = 0, \tag{3}$$

where

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

$$H_{\mu\nu} := 2\left[RR_{\mu\nu} - 2R_{\alpha\mu}R^\alpha_\nu - 2R^{\alpha\beta}R_{\mu\alpha\nu\beta} + R^{\alpha\beta\gamma}_\mu R_{\nu\alpha\beta\gamma} \right] - \frac{1}{2} g_{\mu\nu} L_2$$

and

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$$

is the energy-momentum tensor and $T = g^{\mu\nu}T_{\mu\nu}$. Note that we have the Einstein equations if $\alpha = 0$.

Spherically symmetric spacetimes in n-dimension Globally hyperbolic spacetimes M with n-dimensional spherical symmetry imply that the group $SO(n-1)$ acts by isometry on M and preserves Ψ. We assume

$$Q = M/SO(n-1)$$

[1] Our notation follows the text book by Hawking and Ellis [HE].

inherits from spacetime metric g the structure of a $1+1$-dimensional Lorentzian manifold with boundary with metric $\tilde{g}$, such that

$$\begin{aligned} g &= \tilde{g} + r^2 d\sigma^2, \\ &= -\Omega^2 du dv + r^2 d\sigma^2, \end{aligned} \tag{4}$$

where $d\sigma^2$ is the standard metric of $(n-2)$-sphere. Functions Ω and r depend on only u and v on Q. The boundary of Q consists of $\Gamma \cap S$, where Γ is a connected timelike curve and S is a connected spacelike curve. $\Gamma \cap S$ is a single point and $r(p) = 0$ if and only if $p \in \Gamma$. Γ is called the *centre*. It is assumed that Q is foliated by connected constant v-segments (called ingoing) with past endpoint on S and by connected constant u-segments (outgoing) with past endpoint on $\Gamma \cap S$. The curve S has a unique limit point i^0 on $\overline{Q} \setminus Q$, which is called *spatial infinity*. Let $\mathcal{U}$ be the set of all u defined by

$$\mathcal{U} := \{u | \sup_{v:(u,v)\in Q} r(u,v) = \infty\}.$$

For each $u \in \mathcal{U}$, there is a unique $v^*(u)$ such that $(u, v^*(u)) \in \overline{Q} \setminus Q^+$. Define the *future null infinity* $\mathcal{I}^+$ as follows:

$$\mathcal{I}^+ := \bigcup_{u\in\mathcal{U}} (u, v^*(u)).$$

We will assume that $\mathcal{I}^+ \subset \overline{Q} \setminus Q$ is non-empty.

In this metric, the Einstein equations become as follow:

$$\left[1 + \frac{2\tilde{\alpha}}{r^2}K\right] \partial_u \partial_v r = -\frac{\Omega^2}{4r}\left[(n-3)K + (n-5)\frac{\tilde{\alpha}}{r^2}K^2\right] + \frac{\kappa_n^2}{n-2} r T_{uv}, \tag{5}$$

$$\begin{aligned} \left[1 + \frac{2\tilde{\alpha}}{r^2}K\right] \partial_u \partial_v \log\Omega &= \frac{(n-3)}{r^2}\partial_u r \partial_v r + \frac{k(n-3)}{4r^2}\Omega^2 - \frac{(n-4)r}{2r^2}\partial_u\partial_v r \\ &\quad + \frac{\tilde{\alpha}\Omega^2}{2r^4}\mathcal{K} + \frac{\Omega^2\kappa_n^2}{24r^2}\left(g^{ab}T_{ab} - 4T^u_u\right), \end{aligned} \tag{6}$$

$$\left[1 + \frac{2\tilde{\alpha}}{r^2}K\right] \partial_u(\Omega^{-2}\partial_u r) = -\frac{\kappa_n^2}{n-2} r\Omega^{-2} T_{uu}, \tag{7}$$

$$\left[1 + \frac{2\tilde{\alpha}}{r^2}K\right] \partial_v(\Omega^{-2}\partial_v r) = -\frac{\kappa_n^2}{n-2} r\Omega^{-2} T_{vv}, \tag{8}$$

Here, $\tilde{\alpha} = (n-3)(n-4)\alpha$ and

$$K \equiv 1 + \frac{4\partial_u r \partial_v r}{\Omega^2},$$

and

$$\mathcal{K} \equiv -\frac{2(n-8)Kr\partial_u\partial_v r}{\Omega^2} - \frac{16r^2}{\Omega^4}\left(\partial_u \ln\Omega\partial_u r\partial_v\partial_v r + \partial_v \ln\Omega\partial_v r\partial_u\partial_u r\right) + (n-5)K^2$$
$$+\frac{8r^2}{\Omega^2}\left(\partial_u\partial_u r\partial_v\partial_v r + 4\partial_u \ln\Omega\partial_v \ln\Omega\partial_u r\partial_v r - (\partial_u\partial_v r)^2\right).$$

A cetral point is called *regular* if $K \sim Cr^2$ holds around the center, where C is a non-zero constant.

Now, we will define the *generalized Misner-Sharp mass* [MN], which is a useful tool to analyze spherical symmetric gravitational system.

$$m = \frac{(n-2)V_{n-2}r^{n-3}}{2\kappa_n^2}\left(K + \frac{\tilde{\alpha}}{r^2}K^2\right), \tag{9}$$

where V_{n-2} is the volume of $(n-2)$-sphere. Evolution of the mass is as follow:

$$\partial_u m = 2r^{n-2}V_{n-2}\Omega^{-2}\left(T_{uv}\partial_u r - T_{uu}\partial_v r\right), \tag{10}$$

and

$$\partial_v m = 2r^{n-2}V_{n-2}\Omega^{-2}\left(T_{uv}\partial_v r - T_{vv}\partial_u r\right). \tag{11}$$

First singularity and trapped regions Let $p \in \overline{Q}$. The indecomposable past subset $J^-(p) \cap Q \subset Q$ is said to be *eventually compactly generated* if there exists a compact subset $X \subset Q$ such that

$$J^-(p) \subset D^+(X) \cup J^-(X). \tag{12}$$

Here, the *causal future* (*causal past*) of $p \in M$, denoted $J^+(p)$ ($J^-(p)$), is defined as the set of events that can be reached by a future (past) directed causal curve starting from p and *future (past) Cauchy development* of X is defined as the set of all points $p \in Q$ such that every past-(future-)inextendible non-spacelike curve through p intersects X. A point $p \in \overline{Q} \setminus Q$ is said to be a *first singularity* if $J^-(p) \cap Q$ is eventually compactly generated and if any eventually compactly generated indecomposable proper subset of $J^-(p) \cap Q$ is of the form $J^-(q)$ for a $q \in Q$.

Now, we define the following three regions:

- *Regular region*: $\mathcal{R} = \{q \in Q : \partial_v r > 0, \partial_u r < 0\}$,
- *Trapped region*: $\mathcal{T} = \{q \in Q : \partial_v r < 0, \partial_u r < 0\}$,
- *Marginally trapped region*: $\mathcal{A} = \{q \in Q : \partial_v r = 0, \partial_u r < 0\}$.

In addition, we call $\mathcal{R} \cup \mathcal{A}$ the *non-trapped* region.

Main results

Theorem 2 *Let $p \in \overline{Q} \setminus Q$ be a first singularity. Then either*

$$p \in \overline{\Gamma} \setminus \Gamma$$

or

$$J^{-}(p) \cap Q \cap D^{+}(X) \cap \mathcal{T} \neq \emptyset,$$

for all compact X satisfying (12).

Theorem 3 *$\mathcal{I}^{+}$ is future complete.*

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