Approximation theorem for evolution equations of hyperbolic type in Hilbert space

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Let $X$ be a (complex) Hilbert space. Let $\{A(t); 0 \leq t \leq T\}$ be a family of closed linear operators in $X$. We are concerned with the linear evolution equation

$$
\begin{align*}
\frac{du(t)}{dt} + A(t)u(t) &= f(t) \quad \text{on } [0, T], \\
u(0) &= u_0.
\end{align*}
$$

Let $S$ be a selfadjoint operator with domain $D(S)$ in $X$, satisfying $(u,Su) \geq \|u\|^2$ for $u \in D(S)$. Then the square root $S^{1/2}$ is well-defined and its domain $Y := D(S^{1/2})$ is regarded as a Hilbert space with inner product $(S^{1/2}u,S^{1/2}v)$. In what follows we denote by $B(Y,X)$ the set of all bounded linear operators on $Y$ to $X$. Assume that $A(t)$ satisfies the conditions (I), (II) and (III) stated in [1] or [2]. The solvability of $(E)$ is proved in [2] under the condition that $A(t) \in C([0,T]; B(Y,X))$.

In this paper, we consider the approximate problem to $(E)$:

$$
\begin{align*}
\frac{du_\varepsilon(t)}{dt} + A_\varepsilon(t)u_\varepsilon(t) &= f(t) \quad \text{on } (0, T], \\
u_\varepsilon(0) &= u_0,
\end{align*}
$$

where $A_\varepsilon(t) := A(t) + \varepsilon S$. Under conditions (I)–(III), $-A_\varepsilon(t)$ is a generator of an analytic semigroup on $X$. In this sense, $(E_\varepsilon)$ is an equation of parabolic type, while $(E)$ is of hyperbolic type. Namely, $(E_\varepsilon)$ is a parabolic regularization of the hyperbolic problem $(E)$. We have tried to make it clear that each solution to $(E)$ is a limit of the corresponding family of solutions to $(E_\varepsilon)$. Thus we have the following

**Theorem.** Assume that there are two constants $1/2 < \alpha \leq 1$, $0 < \beta \leq 1$ such that $A(\cdot) \in C^{0,\alpha}([0,T]; B(Y,X))$ and $f(\cdot) \in C^{0,\beta}([0,T]; X) \cap L^1(0,T; Y)$. Let $u(t)$ be a solution to $(E)$. Then the solutions $u_\varepsilon(t)$ to $(E_\varepsilon)$ exist and converge to $u(t)$, i.e.,

$$
u_\varepsilon(t) \to u(t) \quad (\varepsilon \downarrow 0) \quad \text{in } C([0,T]; X).
$$

Note that $C^{0,\alpha}([0,T]; B(Y,X))$ is the space of all Hölder continuous $B(Y,X)$-valued functions on $[0,T]$. In the Lipschitz case (in which $\alpha = \beta = 1$) the convergence of $u_\varepsilon(t)$ to $u(t)$ is already studied in [1].

**References**
